ARML Competition 2014

Paul J. Karafiol, Head Writer
Paul Dreyer
Edward Early
Zuming Feng
Benji Fisher
Zachary Franco
Chris Jeuell
Winston Luo
Andy Niedermaier
George Reuter
Andy Soffer
Eric Wepsic

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1 Team Problems

Problem 1. There exists a digit Y such that, for any digit X, the seven-digit number $\underline{1}\,\underline{2}\,\underline{3}\,\underline{X}\,\underline{5}\,\underline{Y}\,\underline{7}$ is not a multiple of 11. Compute Y.

Problem 2. A point is selected at random from the interior of a right triangle with legs of length $2\sqrt{3}$ and 4. Let p be the probability that the distance between the point and the nearest vertex is less than 2. Then p can be written in the form $a + \sqrt{b}\pi$, where a and b are rational numbers. Compute (a, b).

Problem 3. The square $ARML$ is contained in the xy-plane with $A = (0, 0)$ and $M = (1, 1)$. Compute the length of the shortest path from the point $(2/7, 3/7)$ to itself that touches three of the four sides of square $ARML$.

Problem 4. For each positive integer k, let S_k denote the infinite arithmetic sequence of integers with first term k and common difference k^2. For example, S_3 is the sequence $3, 12, 21, \ldots$. Compute the sum of all k such that 306 is an element of S_k.

Problem 5. Compute the sum of all values of k for which there exist positive real numbers x and y satisfying the following system of equations.

$$\begin{cases} \log_x y^2 + \log_y x^5 = 2k - 1 \\ \log_{x^2} y^5 - \log_{y^2} x^3 = k - 3 \end{cases}$$

Problem 6. Let $W = (0, 0)$, $A = (7, 0)$, $S = (7, 1)$, and $H = (0, 1)$. Compute the number of ways to tile rectangle $WASH$ with triangles of area $1/2$ and vertices at lattice points on the boundary of $WASH$.

Problem 7. Compute $\sin^2 4^\circ + \sin^2 8^\circ + \sin^2 12^\circ + \cdots + \sin^2 176^\circ$.

Problem 8. Compute the area of the region defined by $x^2 + y^2 \le |x| + |y|$.

Problem 9. The arithmetic sequences $a_1, a_2, a_3, \ldots, a_{20}$ and $b_1, b_2, b_3, \ldots, b_{20}$ consist of 40 distinct positive integers, and $a_{20} + b_{14} = 1000$. Compute the least possible value for $b_{20} + a_{14}$.

Problem 10. Compute the ordered triple (x, y, z) representing the farthest lattice point from the origin that satisfies $xy - z^2 = y^2 z - x = 14$.

2 Answers to Team Problems

Answer 1. 4

Answer 2. $\left(\frac{1}{4}, \frac{1}{27}\right)$

Answer 3. $\frac{2}{7}\sqrt{53}$

Answer 4. 326

Answer 5. $\frac{43}{48}$

Answer 6. 3432

Answer 7. $\frac{45}{2}$

Answer 8. $2+\pi$

Answer 9. 10

Answer 10. $(-266, -3, -28)$

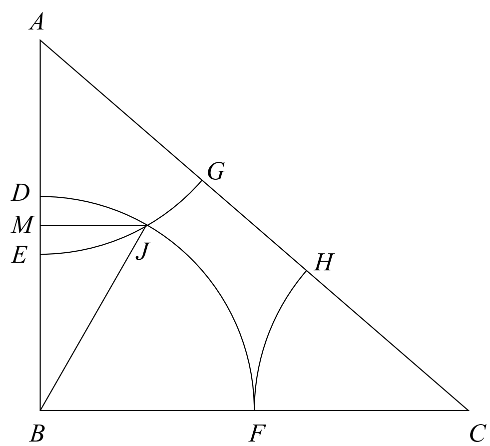
3 Solutions to Team Problems

Problem 1. There exists a digit Y such that, for any digit X, the seven-digit number $\underline{1}\,\underline{2}\,\underline{3}\,\underline{X}\,\underline{5}\,\underline{Y}\,\underline{7}$ is not a multiple of 11. Compute Y.

Solution 1. Consider the ordered pairs of digits (X, Y) for which $\underline{1}\,\underline{2}\,\underline{3}\,\underline{X}\,\underline{5}\,\underline{Y}\,\underline{7}$ **is** a multiple of 11. Recall that a number is a multiple of 11 if and only if the alternating sum of the digits is a multiple of 11. Because $1 + 3 + 5 + 7 = 16$, the sum of the remaining digits, namely $2 + X + Y$, must equal 5 or 16. Thus $X + Y$ must be either 3 or 14, making $X = 3 - Y$ (if $Y = 0$, 1, 2, or 3) or $14 - Y$ (if $Y = 5$, 6, 7, 8, or 9). Thus a solution (X, Y) exists unless $Y = \mathbf{4}$.

Problem 2. A point is selected at random from the interior of a right triangle with legs of length $2\sqrt{3}$ and 4. Let p be the probability that the distance between the point and the nearest vertex is less than 2. Then p can be written in the form $a + \sqrt{b}\pi$, where a and b are rational numbers. Compute (a, b).

Solution 2. Label the triangle as $\triangle ABC$, with $AB = 2\sqrt{3}$ and $BC = 4$. Let D and E lie on $\overline{AB}$ such that $DB = AE = 2$. Let F be the midpoint of $\overline{BC}$, so that $BF = FC = 2$. Let G and H lie on $\overline{AC}$, with $AG = HC = 2$. Now draw the arcs of radius 2 between E and G, D and F, and F and H. Let the intersection of arc DF and arc EG be J. Finally, let M be the midpoint of $\overline{AB}$. The completed diagram is shown below.

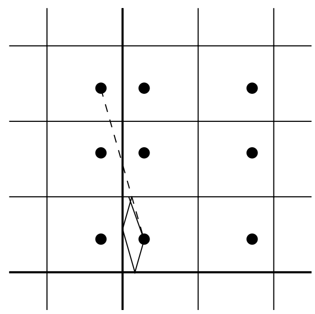


The region R consisting of all points within $\triangle ABC$ that lie within 2 units of any vertex is the union of the three sectors EAG, DBF, and FCH. The angles of these sectors, being the angles $\angle A, \angle B$, and $\angle C$, sum to $180°$, so the sum of their areas is 2π. Computing the area of R requires subtracting the areas of all intersections of the three sectors that make up R.
The only sectors that intersect are EAG and DBF. Half this area of intersection, the part above $\overline{MJ}$, equals the difference between the areas of sector DBJ and of $\triangle MBJ$. Triangle MBJ is a $1 : \sqrt{3} : 2$ right triangle because $BM = \sqrt{3}$ and $BJ = 2$, so the area of $\triangle MBJ$ is $\frac{\sqrt{3}}{2}$. Sector DBJ has area $\frac{1}{12}(4\pi) = \frac{\pi}{3}$, because $m\angle DBJ = 30°$. Therefore the area of intersection of the sectors is $2(\frac{\pi}{3} - \frac{\sqrt{3}}{2}) = \frac{2\pi}{3} - \sqrt{3}$. Hence the total area of R is $2\pi - (\frac{2\pi}{3} - \sqrt{3}) = \frac{4\pi}{3} + \sqrt{3}$.

The total area of $\triangle ABC$ is $4\sqrt{3}$, therefore the desired probability is $\frac{\frac{4\pi}{3}+\sqrt{3}}{4\sqrt{3}} = \frac{\pi}{3\sqrt{3}} + \frac{1}{4}$. Then $a = \frac{1}{4}$ and $b = \left(\frac{1}{3\sqrt{3}}\right)^2 = \frac{1}{27}$, hence the answer is $\left(\mathbf{\frac{1}{4}, \frac{1}{27}}\right)$.

Problem 3. The square $ARML$ is contained in the xy-plane with $A = (0,0)$ and $M = (1,1)$. Compute the length of the shortest path from the point $(2/7, 3/7)$ to itself that touches three of the four sides of square $ARML$.

Solution 3. Consider repeatedly reflecting square $ARML$ over its sides so that the entire plane is covered by copies of $ARML$. A path starting at $(2/7, 3/7)$ that touches one or more sides and returns to $(2/7, 3/7)$ corresponds to a straight line starting at $(2/7, 3/7)$ and ending at the image of $(2/7, 3/7)$ in one of the copies of $ARML$. To touch three sides, the path must cross three lines, at least one of which must be vertical and at least one of which must be horizontal.



If the path crosses two horizontal lines and the line $x = 0$, it will have traveled a distance of 2 units vertically and 4/7 units vertically for a total distance of $\sqrt{2^2 + (4/7)^2}$ units. Similarly, the total distance traveled when crossing two horizontal lines and $x = 1$ is $\sqrt{2^2 + (10/7)^2}$, the total distance traveled when crossing two vertical lines and $y = 0$ is $\sqrt{2^2 + (6/7)^2}$, and the total distance traveled when crossing two vertical lines and $y = 1$ is $\sqrt{2^2 + (8/7)^2}$. The least of these is

$$\sqrt{2^2 + (4/7)^2} = \mathbf{\frac{2}{7}\sqrt{53}}.$$

Problem 4. For each positive integer k, let S_k denote the infinite arithmetic sequence of integers with first term k and common difference k^2. For example, S_3 is the sequence $3, 12, 21, \ldots$. Compute the sum of all k such that 306 is an element of S_k.

Solution 4. If 306 is an element of S_k, then there exists an integer $m \geq 0$ such that $306 = k + mk^2$. Thus $k \mid 306$ and $k^2 \mid 306 - k$. The second relation can be rewritten as $k \mid 306/k - 1$, which implies that $k \leq \sqrt{306}$ unless $k = 306$. The prime factorization of 306 is $2 \cdot 3^2 \cdot 17$, so the set of factors of

306 less than $\sqrt{306}$ is $\{1, 2, 3, 6, 9, 17\}$. Check each in turn:

$$\begin{aligned} 306 - 1 &= 305, & 1^2 &\mid 305 \\ 306 - 2 &= 304, & 2^2 &\mid 304 \\ 306 - 3 &= 303, & 3^2 &\nmid 303 \\ 306 - 6 &= 300, & 6^2 &\nmid 300 \\ 306 - 9 &= 297, & 9^2 &\nmid 297 \\ 306 - 17 &= 289, & 17^2 &\mid 289. \end{aligned}$$

Thus the set of possible k is $\{1, 2, 17, 306\}$, and the sum is $1 + 2 + 17 + 306 = \mathbf{326}$.

Problem 5. Compute the sum of all values of k for which there exist positive real numbers x and y satisfying the following system of equations.

$$\begin{cases} \log_x y^2 + \log_y x^5 &= 2k - 1 \\ \log_{x^2} y^5 - \log_{y^2} x^3 &= k - 3 \end{cases}$$

Solution 5. Let $\log_x y = a$. Then the first equation is equivalent to $2a + \frac{5}{a} = 2k - 1$, and the second equation is equivalent to $\frac{5a}{2} - \frac{3}{2a} = k - 3$. Solving this system by eliminating k yields the quadratic equation $3a^2 + 5a - 8 = 0$, hence $a = 1$ or $a = -\frac{8}{3}$. Substituting each of these values of a into either of the original equations and solving for k yields $(a, k) = (1, 4)$ or $\left(-\frac{8}{3}, -\frac{149}{48}\right)$. Adding the values of k yields the answer of $\mathbf{43/48}$.

Alternate Solution: In terms of $a = \log_x y$, the two equations become $2a + \frac{5}{a} = 2k - 1$ and $\frac{5a}{2} - \frac{3}{2a} = k - 3$. Eliminate $\frac{1}{a}$ to obtain $31a = 16k - 33$; substitute this into either of the original equations and clear denominators to get $96k^2 - 86k - 1192 = 0$. The sum of the two roots is $86/96 = \mathbf{43/48}$.

Problem 6. Let $W = (0, 0)$, $A = (7, 0)$, $S = (7, 1)$, and $H = (0, 1)$. Compute the number of ways to tile rectangle $WASH$ with triangles of area $1/2$ and vertices at lattice points on the boundary of $WASH$.

Solution 6. Define a *fault line* to be a side of a tile other than its base. Any tiling of $WASH$ can be represented as a sequence of tiles $t_1, t_2, \ldots, t_{14}$, where t_1 has a fault line of $\overline{WH}$, t_{14} has a fault line of $\overline{AS}$, and where t_k and t_{k+1} share a fault line for $1 \leq k \leq 13$. Also note that to determine the position of tile t_{k+1}, it is necessary and sufficient to know the fault line that t_{k+1} shares with t_k, as well as whether the base of t_{k+1} lies on $\overline{WA}$ (abbreviated "B" for "bottom") or on $\overline{SH}$ (abbreviated "T" for "top"). Because rectangle $WASH$ has width 7, precisely 7 of the 14 tiles must have their bases on $\overline{WA}$. Thus any permutation of 7 B's and 7 T's determines a unique tiling $t_1, t_2, \ldots, t_{14}$, and conversely, any tiling $t_1, t_2, \ldots, t_{14}$ corresponds to a unique permutation of 7 B's and 7 T's. Thus the answer is $\binom{14}{7} = \mathbf{3432}$.

Alternate Solution: Let $T(a, b)$ denote the number of ways to triangulate the polygon with vertices at $(0, 0)$, $(b, 0)$, $(a, 1)$, $(0, 1)$, where each triangle has area $1/2$ and vertices at lattice points. The problem is to compute $T(7, 7)$. It is easy to see that $T(a, 0) = T(0, b) = 1$ for all a and b. If a and b

are both positive, then either one of the triangles includes the edge from $(a-1,1)$ to $(b,0)$ or one of the triangles includes the edge from $(a,1)$ to $(b-1,0)$, but not both. (In fact, as soon as there is an edge from $(a,1)$ to $(x,0)$ with $x<b$, there must be edges from $(a,1)$ to $(x',0)$ for all $x \le x' < b$.) If there is an edge from $(a-1,1)$ to $(b,0)$, then the number of ways to complete the triangulation is $T(a-1,b)$; if there is an edge from $(a,1)$ to $(b-1,0)$, then the number of ways to complete the triangulation is $T(a,b-1)$; thus $T(a,b) = T(a-1,b) + T(a,b-1)$. The recursion and the initial conditions describe Pascal's triangle, so $T(a,b) = \binom{a+b}{a}$. In particular, $T(7,7) = \binom{14}{7} = \mathbf{3432}$.

Problem 7. Compute $\sin^2 4^\circ + \sin^2 8^\circ + \sin^2 12^\circ + \cdots + \sin^2 176^\circ$.

Solution 7. Because $\cos 2x = 1 - 2\sin^2 x$, $\sin^2 x = \frac{1-\cos 2x}{2}$. Thus the desired sum can be rewritten as

$$\frac{1-\cos 8^\circ}{2} + \frac{1-\cos 16^\circ}{2} + \cdots + \frac{1-\cos 352^\circ}{2} = \frac{44}{2} - \frac{1}{2}\left(\cos 8^\circ + \cos 16^\circ + \cdots + \cos 352^\circ\right).$$

If $\alpha = \cos 8^\circ + i \sin 8^\circ$, then α is a primitive 45^{th} root of unity, and $1 + \alpha + \alpha^2 + \alpha^3 + \cdots + \alpha^{44} = 0$. Hence $\alpha + \alpha^2 + \cdots + \alpha^{44} = -1$, and because the real part of α^n is simply $\cos 8n^\circ$,

$$\cos 8^\circ + \cos 16^\circ + \cdots + \cos 352^\circ = -1.$$

Thus the desired sum is $22 - (1/2)(-1) = \mathbf{45/2}$.

Alternate Solution: The problem asks to simplify the sum

$$\sin^2 a + \sin^2 2a + \sin^2 3a + \cdots + \sin^2 na$$

where $a = 4^\circ$ and $n = 44$. Because $\cos 2x = 1 - 2\sin^2 x$, $\sin^2 x = \frac{1-\cos 2x}{2}$. Thus the desired sum can be rewritten as

$$\frac{1-\cos 2a}{2} + \frac{1-\cos 4a}{2} + \cdots + \frac{1-\cos 2na}{2} = \frac{n}{2} - \frac{1}{2}\left(\cos 2a + \cos 4a + \cdots + \cos 2na\right).$$

Let $Q = \cos 2a + \cos 4a + \cdots + \cos 2na$. By the sum-to-product identity,

$$\begin{aligned} \sin 3a - \sin a &= 2\cos 2a \sin a, \\ \sin 5a - \sin 3a &= 2\cos 4a \sin a, \\ &\vdots \\ \sin(2n+1)a - \sin(2n-1)a &= 2\cos 2na \sin a. \end{aligned}$$

Thus

$$\begin{aligned} Q \cdot 2\sin a &= (\sin 3a - \sin a) + (\sin 5a - \sin 3a) + \cdots + (\sin(2n+1)a - \sin(2n-1)a) \\ &= \sin(2n+1)a - \sin a. \end{aligned}$$

With $a = 4^\circ$ and $n = 44$, the difference on the right side becomes $\sin 356^\circ - \sin 4^\circ$; note that the terms in this difference are opposites, because of the symmetry of the unit circle. Hence

$$\begin{aligned} Q \cdot 2\sin 4^\circ &= -2\sin 4^\circ, \text{ and} \\ Q &= -1. \end{aligned}$$

Thus the original sum becomes $44/2 - (1/2)(-1) = \mathbf{45/2}$.

Problem 8. Compute the area of the region defined by $x^2 + y^2 \leq |x| + |y|$.

Solution 8. Call the region R, and let R_q be the portion of R in the q^{th} quadrant. Noting that the point (x, y) is in R if and only if $(\pm x, \pm y)$ is in R, it follows that $[R_1] = [R_2] = [R_3] = [R_4]$, and so $[R] = 4[R_1]$. So it suffices to determine $[R_1]$.

In the first quadrant, the boundary equation is just $x^2 + y^2 = x + y \Rightarrow (x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 = \frac{1}{2}$. This equation describes a circle of radius $\frac{\sqrt{2}}{2}$ centered at $(\frac{1}{2}, \frac{1}{2})$. The portion of the circle's interior which is inside the first quadrant can be decomposed into a right isosceles triangle with side length 1 and half a circle of radius $\frac{\sqrt{2}}{2}$. Thus $[R_1] = \frac{1}{2} + \frac{\pi}{4}$, hence $[R] = \mathbf{2 + \pi}$.

Problem 9. The arithmetic sequences $a_1, a_2, a_3, \ldots, a_{20}$ and $b_1, b_2, b_3, \ldots, b_{20}$ consist of 40 distinct positive integers, and $a_{20} + b_{14} = 1000$. Compute the least possible value for $b_{20} + a_{14}$.

Solution 9. Write $a_n = a_1 + r(n-1)$ and $b_n = b_1 + s(n-1)$. Then $a_{20} + b_{14} = a_1 + b_1 + 19r + 13s$, while $b_{20} + a_{14} = a_1 + b_1 + 13r + 19s = a_{20} + b_{14} + 6(s - r)$. Because both sequences consist only of integers, r and s must be integers, so $b_{20} + a_{14} \equiv a_{20} + b_{14} \bmod 6$. Thus the least possible value of $b_{20} + a_{14}$ is 4. If $b_{20} = 3$ and $a_{14} = 1$, then $\{a_n\}$ must be a decreasing sequence (else a_{13} would not be positive) and $a_{20} \leq -5$, which is impossible. The case $b_{20} = a_{14} = 2$ violates the requirement that the terms be distinct, and by reasoning analogous to the first case, $b_{20} = 1, a_{14} = 3$ is also impossible. Hence the sum $b_{20} + a_{14}$ is at least 10. To show that 10 is attainable, make $\{a_n\}$ decreasing and b_{20} as small as possible: set $b_{20} = 1$, $a_{14} = 9$, and $a_n = 23 - n$. Then $a_{20} = 3$, yielding $b_{14} = 997$. Hence $s = \frac{997-1}{14-20} = \frac{996}{-6} = -166$ and $b_1 = 997 - (13)(-166) = 3155$, yielding $b_n = 3155 - 166(n-1)$. Because $b_{20} = 1 \leq a_{20}$ and $b_{19} = 167 \geq a_1$, the sequences $\{b_n\}$ and $\{a_n\}$ are distinct for $1 \leq n \leq 20$, completing the proof. Hence the minimum possible value of $b_{20} + a_{14}$ is **10**. [Note: This solution, which improves on the authors' original solution, is due to Ravi Jagadeesan of Phillips Exeter Academy.]

Problem 10. Compute the ordered triple (x, y, z) representing the farthest lattice point from the origin that satisfies $xy - z^2 = y^2 z - x = 14$.

Solution 10. First, eliminate x: $y(y^2 z - x) + (xy - z^2) = 14(y+1) \Rightarrow z^2 - y^3 z + 14(y+1) = 0$. Viewed as a quadratic in z, this equation implies $z = \frac{y^3 \pm \sqrt{y^6 - 56(y+1)}}{2}$. In order for z to be an integer, the discriminant must be a perfect square. Because $y^6 = (y^3)^2$ and $(y^3 - 1)^2 = y^6 - 2y^3 + 1$, it follows that $|56(y+1)| \geq 2|y^3| - 1$. This inequality only holds for $|y| \leq 5$. Within that range, the only values of y for which $y^6 - 56y - 56$ is a perfect square are -1 and -3. If $y = -1$, then $z = -1$ or $z = 0$. If $y = -3$, then $z = 1$ or $z = -28$. After solving for the respective values of x in the various cases, the four lattice points satisfying the system are $(-15, -1, -1), (-14, -1, 0), (-5, -3, 1)$, and $(-266, -3, -28)$. The farthest solution point from the origin is therefore $\mathbf{(-266, -3, -28)}$.

4 Power Question 2014: Power of Potlucks

Instructions: The power question is worth 50 points; each part's point value is given in brackets next to the part. To receive full credit, the presentation must be legible, orderly, clear, and concise. If a problem says "list" or "compute," you need not justify your answer. If a problem says "determine," "find," or "show," then you must show your work or explain your reasoning to receive full credit, although such explanations do not have to be lengthy. If a problem says "justify" or "prove," then you must prove your answer rigorously. Even if not proved, earlier numbered items may be used in solutions to later numbered items, but not vice versa. Pages submitted for credit should be NUMBERED IN CONSECUTIVE ORDER AT THE TOP OF EACH PAGE in what your team considers to be proper sequential order. PLEASE WRITE ON ONLY ONE SIDE OF THE ANSWER PAPERS. Put the TEAM NUMBER (not the team name) on the cover sheet used as the first page of the papers submitted. Do not identify the team in any other way.

In each town in ARMLandia, the residents have formed groups, which meet each week to share math problems and enjoy each others' company over a potluck-style dinner. Each town resident belongs to exactly one group. Every week, each resident is required to make one dish and to bring it to his/her group.

It so happens that each resident knows how to make precisely two dishes. Moreover, no two residents of a town know how to make the same pair of dishes. Shown below are two example towns. In the left column are the names of the town's residents. Adjacent to each name is the list of dishes that the corresponding resident knows how to make.

ARMLton

Resident	Dishes
Paul	pie, turkey
Arnold	pie, salad
Kelly	salad, broth

ARMLville

Resident	Dishes
Sally	steak, calzones
Ross	calzones, pancakes
David	steak, pancakes

The *population* of a town T, denoted $\operatorname{pop}(T)$, is the number of residents of T. Formally, the town itself is simply the set of its residents, denoted by $\{r_1, \ldots, r_{\operatorname{pop}(T)}\}$ unless otherwise specified. The set of dishes that the residents of T collectively know how to make is denoted $\operatorname{dish}(T)$. For example, in the town of ARMLton described above, $\operatorname{pop}(\text{ARMLton}) = 3$, and $\operatorname{dish}(\text{ARMLton}) = \{\text{pie, turkey, salad, broth}\}$.

A town T is called *full* if for every pair of dishes in $\operatorname{dish}(T)$, there is exactly one resident in T who knows how to make those two dishes. In the examples above, ARMLville is a full town, but ARMLton is not, because (for example) nobody in ARMLton knows how to make both turkey and salad.

Denote by $\mathcal{F}_d$ a full town in which collectively the residents know how to make d dishes. That is, $|\operatorname{dish}(\mathcal{F}_d)| = d$.

1a. Compute $\operatorname{pop}(\mathcal{F}_{17})$. [1 pt]

1b. Let $n = \operatorname{pop}(\mathcal{F}_d)$. In terms of n, compute d. [1 pt]

1c. Let T be a full town and let $D \in \operatorname{dish}(T)$. Let T' be the town consisting of all residents of T who do not know how to make D. Prove that T' is full. [2 pts]

In order to avoid the embarrassing situation where two people bring the same dish to a group dinner, if two people know how to make a common dish, they are forbidden from participating in the same group meeting. Formally, a *group assignment* on T is a function $f : T \to \{1, 2, \ldots, k\}$, satisfying the condition that if $f(r_i) = f(r_j)$ for $i \neq j$, then r_i and r_j do not know any of the same recipes. The *group number* of a town T, denoted $\operatorname{gr}(T)$, is the least positive integer k for which there exists a group assignment on T.

For example, consider once again the town of ARMLton. A valid group assignment would be $f(\text{Paul}) = f(\text{Kelly}) = 1$ and $f(\text{Arnold}) = 2$. The function which gives the value 1 to each resident of ARMLton is **not** a group assignment, because Paul and Arnold must be assigned to different groups.

2a. Show that $\text{gr}(\text{ARMLton}) = 2$. [1 pt]

2b. Show that $\text{gr}(\text{ARMLville}) = 3$. [1 pt]

3a. Show that $\text{gr}(\mathcal{F}_4) = 3$. [1 pt]

3b. Show that $\text{gr}(\mathcal{F}_5) = 5$. [2 pts]

3c. Show that $\text{gr}(\mathcal{F}_6) = 5$. [2 pts]

4. Prove that the sequence $\text{gr}(\mathcal{F}_2), \text{gr}(\mathcal{F}_3), \text{gr}(\mathcal{F}_4), \ldots$ is a non-decreasing sequence. [2 pts]

For a dish D, a resident is called a *D-chef* if he or she knows how to make the dish D. Define $\text{chef}_T(D)$ to be the set of residents in T who are D-chefs. For example, in ARMLville, David is a steak-chef and a pancakes-chef. Further, $\text{chef}_{\text{ARMLville}}(\text{steak}) = \{\text{Sally, David}\}$.

5. Prove that

$$\sum_{D \in \text{dish}(T)} |\text{chef}_T(D)| = 2\,\text{pop}(T).$$ [2 pts]

6. Show that for any town T and any $D \in \text{dish}(T)$, $\text{gr}(T) \geq |\text{chef}_T(D)|$. [2 pts]

If $\text{gr}(T) = |\text{chef}_T(D)|$ for some $D \in \text{dish}(T)$, then T is called *homogeneous*. If $\text{gr}(T) > |\text{chef}_T(D)|$ for each dish $D \in \text{dish}(T)$, then T is called *heterogeneous*. For example, ARMLton is homogeneous, because $\text{gr}(\text{ARMLton}) = 2$ and exactly two chefs make pie, but ARMLville is heterogeneous, because even though each dish is only cooked by two chefs, $\text{gr}(\text{ARMLville}) = 3$.

7. For $n = 5, 6,$ and 7, find a heterogeneous town T of population 5 for which $|\text{dish}(T)| = n$. [3 pts]

A *resident cycle* is a sequence of distinct residents $r_1, \ldots, r_n$ such that for each $1 \leq i \leq n-1$, the residents r_i and r_{i+1} know how to make a common dish, residents r_n and r_1 know how to make a common dish, and no other pair of residents r_i and r_j, $1 \leq i, j \leq n$ know how to make a common dish. Two resident cycles are *indistinguishable* if they contain the same residents (in any order), and distinguishable otherwise. For example, if r_1, r_2, r_3, r_4 is a resident cycle, then r_2, r_1, r_4, r_3 and r_3, r_2, r_1, r_4 are indistinguishable resident cycles.

8a. Compute the number of distinguishable resident cycles of length 6 in $\mathcal{F}_8$. [1 pt]

8b. In terms of k and d, find the number of distinguishable resident cycles of length k in $\mathcal{F}_d$. [1 pt]

9. Let T be a town with at least two residents that has a single resident cycle that contains every resident. Prove that T is homogeneous if and only if $\text{pop}(T)$ is even. [3 pts]

10. Let T be a town such that, for each $D \in \text{dish}(T)$, $|\text{chef}_T(D)| = 2$.

- **a.** Prove that there are finitely many resident cycles $C_1, C_2, \ldots, C_j$ in T so that each resident belongs to exactly one of the C_i. [3 pts]
- **b.** Prove that if $\text{pop}(T)$ is odd, then T is heterogeneous. [3 pts]

11. Let T be a town such that, for each $D \in \text{dish}(T)$, $|\text{chef}_T(D)| = 3$.

- **a.** Either find such a town T for which $|\text{dish}(T)|$ is odd, or show that no such town exists. [2 pts]
- **b.** Prove that if T contains a resident cycle such that for every dish $D \in \text{dish}(T)$, there exists a chef in the cycle that can prepare D, then $\text{gr}(T) = 3$. [3 pts]

12. Let k be a positive integer, and let T be a town in which $|\text{chef}_T(D)| = k$ for every dish $D \in \text{dish}(T)$. Suppose further that $|\text{dish}(T)|$ is odd.

- **a.** Show that k is even. [1 pt]
- **b.** Prove the following: for every group in T, there is some dish $D \in \text{dish}(T)$ such that no one in the group is a D-chef. [3 pts]
- **c.** Prove that $\text{gr}(T) > k$. [3 pts]

13a. For each odd positive integer $d \geq 3$, prove that $\text{gr}(\mathcal{F}_d) = d$. [3 pts]

13b. For each even positive integer d, prove that $\text{gr}(\mathcal{F}_d) = d - 1$. [4 pts]

5 Solutions to Power Question

1. **a.** There are $\binom{17}{2} = 136$ possible pairs of dishes, so $\mathcal{F}_{17}$ must have 136 people.

b. With d dishes there are $\binom{d}{2} = \frac{d^2-d}{2}$ possible pairs, so $n = \frac{d^2-d}{2}$. Then $2n = d^2 - d$, or $d^2 - d - 2n = 0$. Using the quadratic formula yields $d = \frac{1+\sqrt{1+8n}}{2}$ (ignoring the negative value).

c. The town T' consists of all residents of T who do not know how to make D. Because T is full, every pair of dishes $\{d_i, d_j\}$ in $\text{dish}(T)$ can be made by some resident r_{ij} in T. If $d_i \neq D$ and $d_j \neq D$, then $r_{ij} \in T'$. So every pair of dishes in $\text{dish}(T) \setminus \{D\}$ can be made by some resident of T'. Hence T' is full.

2. **a.** Paul and Arnold cannot be in the same group, because they both make pie, and Arnold and Kelly cannot be in the same group, because they both make salad. Hence there must be at least two groups. But Paul and Kelly make none of the same dishes, so they can be in the same group. Thus a valid group assignment is

$$\begin{aligned} \text{Paul} &\mapsto 1 \\ \text{Kelly} &\mapsto 1 \\ \text{Arnold} &\mapsto 2. \end{aligned}$$

Hence $\text{gr}(\text{ARMLton}) = 2$.

b. Sally and Ross both make calzones, Ross and David both make pancakes, and Sally and David both make steak. So no two of these people can be in the same group, and $\text{gr}(\text{ARMLville}) = 3$.

3. **a.** Let the dishes be d_1, d_2, d_3, d_4 and let resident r_{ij} make dishes d_i and d_j, where $i < j$. There are six pairs of dishes, which can be divided into nonoverlapping pairs in three ways: $\{1,2\}$ and $\{3,4\}$, $\{1,3\}$ and $\{2,4\}$, and $\{1,4\}$ and $\{2,3\}$. Hence the assignment $r_{12}, r_{34} \mapsto 1$, $r_{13}, r_{24} \mapsto 2$, and $r_{14}, r_{23} \mapsto 3$ is valid, hence $\text{gr}(\mathcal{F}_4) = 3$.

b. First, $\text{gr}(\mathcal{F}_5) \geq 5$: there are $\binom{5}{2} = 10$ people in $\mathcal{F}_5$, and because each person cooks two different dishes, any valid group of three people would require there to be six different dishes—yet there are only five. So each group can have at most two people. A valid assignment using five groups is shown below.

Residents	Group
r_{12}, r_{35}	1
r_{13}, r_{45}	2
r_{14}, r_{23}	3
r_{15}, r_{24}	4
r_{25}, r_{34}	5

c. Now there are $\binom{6}{2} = 15$ people, but there are six different dishes, so it is possible (if done carefully) to place three people in a group. Because four people in a single group would require there to be eight different dishes, no group can have more than three people, and so $15/3 = 5$ groups is minimal. (Alternatively, there are five different residents who can cook dish d_1, and no two of these can be in the same group, so there must be at least five groups.) The assignment below attains that minimum.

Residents	Group
r_{12}, r_{34}, r_{56}	1
r_{13}, r_{25}, r_{46}	2
r_{14}, r_{26}, r_{35}	3
r_{15}, r_{24}, r_{36}	4
r_{16}, r_{23}, r_{45}	5

4. Pick some $n \geq 2$ and a full town $\mathcal{F}_n$ whose residents prepare dishes $d_1, \ldots, d_n$, and let $\text{gr}(\mathcal{F}_n) = k$. Suppose that $f_n : \mathcal{F}_n \to \{1, 2, \ldots, k\}$ is a valid group assignment for $\mathcal{F}_n$. Then remove from $\mathcal{F}_n$ all residents who prepare dish d_n; by problem 1c, this operation yields the full town $\mathcal{F}_{n-1}$. Define $f_{n-1}(r) = f_n(r)$ for each remaining resident r in $\mathcal{F}_n$. If r and s are two (remaining) residents who prepare a common dish, then $f_n(r) \neq f_n(s)$, because f_n was a valid group assignment. Hence $f_{n-1}(r) \neq f_{n-1}(s)$ by construction of f_{n-1}. Therefore f_{n-1} is a valid group assignment on $\mathcal{F}_{n-1}$, and the set of groups to which the residents of $\mathcal{F}_{n-1}$ are assigned is a (not necessarily proper) subset of $\{1, 2, \ldots, k\}$. Thus $\text{gr}(\mathcal{F}_{n-1})$ is at most k, which implies the desired result.

5. Because each chef knows how to prepare exactly two dishes, and no two chefs know how to prepare the same two dishes, each chef is counted exactly twice in the sum $\Sigma\,|\text{chef}_T(D)|$. More formally, consider the set of "resident-dish pairs":

$$S = \{(r, D) \in T \times \text{dish}(T) \mid r \text{ makes } D\}.$$

Count $|S|$ in two different ways. First, every dish D is made by $|\text{chef}_T(D)|$ residents of T, so

$$|S| = \sum_{D \in \text{dish}(T)} |\text{chef}_T(D)|\,.$$

Second, each resident knows how to make exactly two dishes, so

$$|S| = \sum_{r \in T} 2 = 2\,\text{pop}(T).$$

6. Let $D \in \text{dish}(T)$. Suppose that f is a valid group assignment on T. Then for $r, s \in \text{chef}_T(D)$, if $r \neq s$, it follows that $f(r) \neq f(s)$. Hence there must be at least $|\text{chef}_T(D)|$ distinct groups in the range of f, i.e., $\text{gr}(T) \geq |\text{chef}_T(D)|$.

7. For $n = 5$, this result is attained as follows:

Resident	Dishes
Amy	d_1, d_2
Benton	d_2, d_3
Carol	d_3, d_4
Devin	d_4, d_5
Emma	d_5, d_1

For each dish D, note that $\text{chef}_T(D) = 2$. But $\text{gr}(T) > 2$, because if T had at most two groups, at least one of them would contain three people, and choosing any three people will result in a common dish that two of them can cook. Hence T is heterogeneous.

For $n \geq 6$, it suffices to assign dishes to residents so that there are three people who must be in different groups and that no dish is cooked by more than two people, which guarantees that $\text{gr}(T) \geq 3$ and $\text{chef}_T(D) \leq 2$ for each dish D.

Resident	Dishes
Amy	d_1, d_2
Benton	d_1, d_3
Carol	d_2, d_3
Devin	d_4, d_5
Emma	d_5, d_6

Note that Devin's and Emma's dishes are actually irrelevant to the situation, so long as they do not cook any of d_1, d_2, d_3, which already have two chefs each. Thus we can adjust this setup for $n = 7$ by setting Devin's dishes as d_4, d_5 and Emma's dishes as d_6, d_7. (In this last case, Devin and Emma are extremely compatible: they can both be put in a group with anyone else in the town!)

8. **a.** Because the town is full, each pair of dishes is cooked by exactly one resident, so it is simplest to identify residents by the pairs of dishes they cook. Suppose the first resident cooks (d_1, d_2), the second resident (d_2, d_3), the third resident (d_3, d_4), and so on, until the sixth resident, who cooks (d_6, d_1). Then there are 8 choices for d_1 and 7 choices for d_2. There are only 6 choices for d_3, because $d_3 \neq d_1$ (otherwise two residents would cook the same pair of dishes). For $k > 3$, the requirement that no two intermediate residents cook the same dishes implies that d_{k+1} cannot equal any of $d_1, \ldots, d_{k-1}$, and of course d_k and d_{k+1} must be distinct dishes. Hence there are $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 20{,}160$ six-person resident cycles, not accounting for different starting points in the cycle and the two different directions to go around the cycle. Taking these into account, there are $20{,}160/(6 \cdot 2) = 1{,}680$ distinguishable resident cycles.

b. Using the logic from 8a, there are $d(d-1)\cdots(d-k+1)$ choices for $d_1, d_2, \ldots, d_k$. To account for indistinguishable cycles, divide by k possible starting points and 2 possible directions, yielding $\frac{d(d-1)\cdots(d-k+1)}{2k}$ or $\frac{d!}{2k(d-k)!}$ distinguishable resident cycles.

9. Note that for every $D \in \text{dish}(T), \text{chef}_T(D) \leq 2$, because otherwise, $r_1, r_2, \ldots, r_n$ could not be a resident cycle. Without loss of generality, assume the cycle is $r_1, r_2, \ldots, r_n$. If n is even, assign resident r_i to group 1 if i is odd, and to group 2 if i is even. This is a valid group assignment, because the only pairs of residents who cook the same dish are (r_i, r_{i+1}) for $i = 1, 2, \ldots, n-1$ and (r_n, r_1). In each case, the residents are assigned to different groups. This proves $\text{gr}(T) = 2$, so T is homogeneous. On the other hand, if n is odd, suppose for the sake of contradiction that there are only two groups. Then either r_1 and r_n are in the same group, or for some i, r_i and r_{i+1} are in the same group. In either case, two residents in the same group share a dish, contradicting the requirement that no members of a group have a common dish. Hence $\text{gr}(T) \geq 3$ when n is odd, making T heterogeneous.

10. **a.** First note that the condition $|\text{chef}_T(D)| = 2$ for all D implies that $\text{pop}(T) = |\text{dish}(T)|$, using the equation from problem 5. So for the town in question, the population of the town equals the number of dishes in the town. Because no two chefs cook the same pair of dishes, it is impossible for such a town to have exactly two residents, and because each dish is cooked by exactly two chefs, it is impossible for such a town to have only one resident.

The claim is true for towns of three residents satisfying the conditions: such towns must have one resident who cooks dishes d_1 and d_2, one resident who cooks dishes d_2 and d_3, and one resident who cooks dishes d_3 and d_1, and those three residents form a cycle. So proceed by (modified) strong induction: assume that for some $n > 3$ and for all positive integers k such that $3 \leq k < n$, every town T with k residents and $|\text{chef}_T(D)| = 2$ for all $D \in \text{dish}(T)$ can be divided into a finite number of resident cycles such that each resident belongs to exactly one of the cycles. Let T_n be a town of n residents, and arbitrarily pick resident r_1 and dishes d_1 and d_2 cooked by r_1. Then there is exactly one other resident r_2 who also cooks d_2 (because $|\text{chef}_{T_n}(d_2)| = 2$). But r_2 also cooks another dish, d_3, which is cooked by another resident, r_3. Continuing in this fashion, there can be only two outcomes: either the process exhausts all the residents of T_n, or there exists some resident r_m, $m < n$, who cooks the same dishes as r_{m-1} and r_ℓ for $\ell < m - 1$.

In the former case, r_n cooks another dish; but every dish besides d_1 is already cooked by two chefs in T_n, so r_n must also cook d_1, closing the cycle. Because every resident is in this cycle, the statement to be proven is also true for T_n.

In the latter case, the same logic shows that r_m cooks d_1, also closing the cycle, but there are other residents of T_n who have yet to be accounted for. Let $C_1 = \{r_1, \ldots, r_m\}$, and consider the town T' whose residents are $T_n \setminus C_1$. Each of dishes $d_1, \ldots, d_m$ is cooked by two people in C_1, so no chef in T' cooks any of these dishes, and no dish in T' is cooked by any of the people in C_1 (because each person in C_1 already cooks two dishes in the set $\text{dish}(C_1)$). Thus $|\text{chef}_{T'}(D)| = 2$ for each D in $\text{dish}(T')$. It follows that $\text{pop}(T') < \text{pop}(T)$ but $\text{pop}(T') > 0$, so by the inductive hypothesis, the residents of T' can be divided into disjoint resident cycles.

Thus the statement is proved by strong induction.

b. In order for T to be homogeneous, it must be possible to partition the residents into exactly two dining groups. First apply 10a to divide the town into finitely many resident cycles C_i, and assume towards a contradiction that such a group assignment $f : T \to \{1, 2\}$ exists. If $\text{pop}(T)$ is odd, then at least one of the cycles C_i must contain an odd number of residents; without loss of generality, suppose this cycle to be C_1, with residents $r_1, r_2, \ldots, r_{2k+1}$. (By the restrictions noted in part a, $k \geq 1$.) Now because r_i and r_{i+1} cook a dish in common, $f(r_i) \neq f(r_{i+1})$ for all i. Thus if $f(r_1) = 1$, it follows that $f(r_2) = 2$, and that $f(r_3) = 1$, etc. So $f(r_i) = f(r_1)$ if i is odd and $f(r_i) = f(r_2)$ if i is even; in particular, $f(r_{2k+1}) = f(1)$. But that equation would imply that r_1 and r_{2k+1} cook no dishes in common, which is impossible if they are the first and last residents in a resident cycle. So no such group assignment can exist, and $\text{gr}(T) \geq 3$. Hence T is heterogeneous.

11. a. In problem 5, it was shown that

$$2\,\text{pop}(T) = \sum_{D \in \text{dish}(T)} |\text{chef}_T(D)|\,.$$

Therefore $\sum_{D \in \text{dish}(T)} |\text{chef}_T(D)|$ is even. But if $|\text{chef}_T(D)| = 3$ for all $D \in \text{dish}(T)$, then the sum is simply $3\,|\text{dish}(T)|$, so $|\text{dish}(T)|$ must be even.

b. By problem 6, it must be the case that $\text{gr}(T) \geq 3$. Let $C = \{r_1, r_2, \ldots, r_n\}$ denote a resident cycle such that for every dish $D \in \text{dish}(T)$, there exists a chef in C that can prepare D. Each resident is a chef for two dishes, and every dish can be made by two residents in C (although by three in T). Thus the number of residents in the resident cycle C is equal to $|\mathrm{dish}(T)|$, which was proved to be even in the previous part.

Define a group assignment by setting

$$f(r) = \begin{cases} 1 & \text{if } r \notin C \\ 2 & \text{if } r = r_i, i \text{ is even} \\ 3 & \text{if } r = r_i, i \text{ is odd.} \end{cases}$$

For any $D \in \mathrm{dish}(T)$, there are exactly three D-chefs, and exactly two of them belong to the resident cycle C. Hence exactly one of the D-chefs r will have $f(r) = 1$. The remaining two D-chefs will be r_i and r_{i+1} for some i, or r_1 and r_n. In either case, the group assignment f will assign one of them to 2 and the other to 3. Thus any two residents who make a common dish will be assigned different groups by f, so f is a valid group assignment, proving that $\mathrm{gr}(T) = 3$.

12. **a.** From problem 5,

$$2\,\mathrm{pop}(T) = \sum_{D \in \mathrm{dish}(T)} |\mathrm{chef}_T(D)|\,.$$

Because $|\mathrm{chef}_T(D)| = k$ for all $D \in \mathrm{dish}(T)$, the sum is $k \cdot \mathrm{dish}(T)$. Thus $2\,\mathrm{pop}\,T = k \cdot \mathrm{dish}(T)$, and so $k \cdot \mathrm{dish}(T)$ must be even. By assumption, $|\mathrm{dish}(T)|$ is odd, so k must be even.

b. Suppose for the sake of contradiction that there is some n for which the group $R = \{r \in T \mid f(r) = n\}$ has a D-chef for every dish D. Because f is a group assignment and f assigns every resident of R to group n, no two residents of R make the same dish. Thus for every $D \in \mathrm{dish}(T)$, exactly one resident of R is a D-chef; and each D-chef cooks exactly one other dish, which itself is not cooked by anyone else in R. Thus the dishes come in pairs: for each dish D, there is another dish D' cooked by the D-chef in R and no one else in R. However, if the dishes can be paired off, there must be an even number of dishes, contradicting the assumption that $|\mathrm{dish}(T)|$ is odd. Thus for every n, the set $\{r \in T \mid f(r) = n\}$ must be missing a D-chef for some dish D.

c. Let f be a group assignment for T, and let $R = \{r \in T \mid f(r) = 1\}$. From problem 12b, there must be some $D \in \mathrm{dish}(T)$ with no D-chefs in R. Moreover, f cannot assign two D-chefs to the same group, so there must be at least k other groups besides R. Hence there are at least $1 + k$ different groups, so $\mathrm{gr}(T) > k$.

13. **a.** Fix $D \in \mathrm{dish}(\mathcal{F}_d)$. Then for every other dish $D' \in \mathrm{dish}(\mathcal{F}_d)$, there is exactly one chef who makes both D and D', hence $|\mathrm{chef}_{\mathcal{F}_d}(D)| = d-1$, which is even because d is odd. Thus for each $D \in \mathrm{dish}(\mathcal{F}_d)$, $|\mathrm{chef}_{\mathcal{F}_d}(D)|$ is even. Because $|\mathrm{dish}(\mathcal{F}_d)| = d$ is odd and $|\mathrm{chef}_{\mathcal{F}_d}(D)| = d-1$ for every dish in $\mathcal{F}_d$, problem 12c applies, hence $\mathrm{gr}(\mathcal{F}_d) > d-1$.

Label the dishes $D_1, D_2, \ldots, D_d$, and label the residents $r_{i,j}$ for $1 \le i < j \le d$ so that $r_{i,j}$ is a D_i-chef and a D_j-chef. Define $f : \mathcal{F}_d \to \{0, 1, \ldots, d-1\}$ by letting $f(r_{i,j}) \equiv i + j \bmod d$.

Suppose that $f(r_{i,j}) = f(r_{k,\ell})$, so $i + j \equiv k + \ell \bmod d$. Then $r_{i,j}$ and $r_{k,\ell}$ are assigned to the same group, which is a problem if they are different residents but are chefs for the same dish. This overlap occurs if and only if one of i and j is equal to one of k and ℓ. If $i = k$, then $j \equiv \ell \bmod d$. As j and ℓ are both between 1 and d, the only way they could be congruent modulo d is if they were in fact equal. That is, $r_{i,j}$ is the same resident as $r_{k,\ell}$. The other three cases ($i = \ell$, $j = k$, and $j = \ell$) are analogous. Thus f is a valid group assignment, proving that $\mathrm{gr}(\mathcal{F}_d) \le d$. Therefore $\mathrm{gr}(\mathcal{F}_d) = d$.

b. In problem 4, it was shown that the sequence $\operatorname{gr}(\mathcal{F}_2), \operatorname{gr}(\mathcal{F}_3), \ldots$ is nondecreasing. If d is even, $\operatorname{gr}(\mathcal{F}_d) \geq \operatorname{gr}(\mathcal{F}_{d-1})$, and because $d-1$ is odd, problem 13a applies: $\operatorname{gr}(\mathcal{F}_{d-1}) = d-1$. Hence $\operatorname{gr}(\mathcal{F}_d) \geq d-1$. Now it suffices to show that $\operatorname{gr}(\mathcal{F}_d) \leq d-1$ by exhibiting a valid group assignment $f : \mathcal{F}_d \to \{1, 2, \ldots, d-1\}$.

Label the dishes $D_1, \ldots, D_d$, and label the residents $r_{i,j}$ for $1 \leq i < j \leq d$ so that $r_{i,j}$ is a D_i-chef and a D_j-chef. Let $R = \{r_{i,j} \mid i, j \neq d\}$. That is, R is the set of residents who are not D_d-chefs. Using 1c, R is a full town with $d-1$ dishes, so from 12a, it has a group assignment $f : R \to \{1, 2, \ldots, d-1\}$. For each $D_i \in \operatorname{dish}(\mathcal{F}_d)$, $i \neq d$, $|\operatorname{chef}_R(D_i)| = d-2$. Because there are $d-1$ groups and $|\operatorname{chef}_R(D_i)| = d-2$, exactly one group n_i must not contain a D_i-chef for each dish D_i.

It cannot be the case that $n_i = n_j$ for $i \neq j$. Indeed, suppose for the sake of contradiction that $n_i = n_j$. Without loss of generality, assume that $n_i = n_j = 1$ (by perhaps relabeling the dishes). Then any resident $r \in R$ assigned to group 1 (that is, $f(r) = 1$) would be neither a D_i-chef nor a D_j-chef. The residents in R who are assigned to group 1 must all be chefs for the remaining $d-3$ dishes. Because each resident cooks two dishes, and no two residents of group 1 can make a common dish,

$$|\{r \in R \mid f(r) = 1\}| \leq \frac{d-3}{2}.$$

For each of the other groups $2, 3, \ldots, d-1$, the number of residents of R in that group is no more than $(d-1)/2$, because there are $d-1$ dishes in R, each resident cooks two dishes, and no two residents in the same group can make a common dish. However, because $d-1$ is odd, the size of any group is actually no more than $(d-2)/2$. Therefore

$$\begin{aligned}
|R| &= \sum_{k=1}^{d-1} |\{r \in R \mid f(r) = k\}| \\
&= |\{r \in R \mid f(r) = 1\}| + \sum_{k=2}^{d-1} |\{r \in R \mid f(r) = k\}| \\
&\leq \frac{d-3}{2} + \sum_{k=2}^{d-1} \frac{d-2}{2} \\
&= \frac{d-3}{2} + \frac{(d-2)^2}{2} \\
&= \frac{d^2 - 3d + 1}{2} < \frac{d^2 - 3d + 2}{2} = |R|.
\end{aligned}$$

This is a contradiction, so it must be that $n_i \neq n_j$ for all $i \neq j$, making f a valid group assignment on $\mathcal{F}_d$. Hence $\operatorname{gr}(\mathcal{F}_d) = d-1$.

6 Individual Problems

Problem 1. Charlie was born in the twentieth century. On his birthday in the present year (2014), he notices that his current age is twice the number formed by the rightmost two digits of the year in which he was born. Compute the four-digit year in which Charlie was born.

Problem 2. Let A, B, and C be randomly chosen (not necessarily distinct) integers between 0 and 4 inclusive. Pat and Chris compute the value of $A + B \cdot C$ by two different methods. Pat follows the proper order of operations, computing $A+(B\cdot C)$. Chris ignores order of operations, choosing instead to compute $(A + B) \cdot C$. Compute the probability that Pat and Chris get the same answer.

Problem 3. Bobby, Peter, Greg, Cindy, Jan, and Marcia line up for ice cream. In an *acceptable lineup*, Greg is ahead of Peter, Peter is ahead of Bobby, Marcia is ahead of Jan, and Jan is ahead of Cindy. For example, the lineup with Greg in front, followed by Peter, Marcia, Jan, Cindy, and Bobby, in that order, is an acceptable lineup. Compute the number of acceptable lineups.

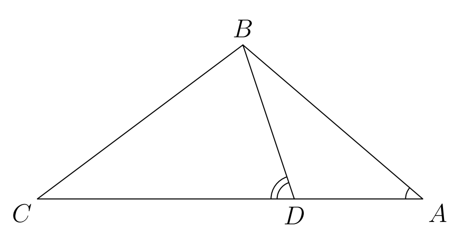
Problem 4. In triangle ABC, $a = 12$, $b = 17$, and $c = 13$. Compute $b\cos C - c\cos B$.

Problem 5. The sequence of words $\{a_n\}$ is defined as follows: $a_1 = X$, $a_2 = O$, and for $n \geq 3$, a_n is a_{n-1} followed by the reverse of a_{n-2}. For example, $a_3 = OX$, $a_4 = OXO$, $a_5 = OXOXO$, and $a_6 = OXOXOOXO$. Compute the number of palindromes in the first 1000 terms of this sequence.

Problem 6. Compute the smallest positive integer n such that $214\cdot n$ and $2014\cdot n$ have the same number of divisors.

Problem 7. Let N be the least integer greater than 20 that is a palindrome in both base 20 and base 14. For example, the three-digit base-14 numeral $(13)5(13)_{14}$ (representing $13 \cdot 14^2 + 5 \cdot 14^1 + 13 \cdot 14^0$) is a palindrome in base 14, but not in base 20, and the three-digit base-14 numeral $(13)31_{14}$ is not a palindrome in base 14. Compute the base-10 representation of N.

Problem 8. In triangle ABC, $BC = 2$. Point D is on $\overline{AC}$ such that $AD = 1$ and $CD = 2$. If $\text{m}\angle BDC = 2\text{m}\angle A$, compute $\sin A$.



Problem 9. Compute the greatest integer $k \leq 1000$ such that $\binom{1000}{k}$ is a multiple of 7.

Problem 10. An integer-valued function f is called *tenuous* if $f(x) + f(y) > x^2$ for all positive integers x and y. Let g be a tenuous function such that $g(1) + g(2) + \cdots + g(20)$ is as small as possible. Compute the minimum possible value for $g(14)$.

7 Answers to Individual Problems

Answer 1. 1938

Answer 2. $\frac{9}{25}$

Answer 3. 20

Answer 4. 10

Answer 5. 667

Answer 6. 19133

Answer 7. 105

Answer 8. $\frac{\sqrt{6}}{4}$

Answer 9. 979

Answer 10. 136

8 Solutions to Individual Problems

Problem 1. Charlie was born in the twentieth century. On his birthday in the present year (2014), he notices that his current age is twice the number formed by the rightmost two digits of the year in which he was born. Compute the four-digit year in which Charlie was born.

Solution 1. Let N be the number formed by the rightmost two digits of the year in which Charlie was born. Then his current age is $100 - N + 14 = 114 - N$. Setting this equal to $2N$ and solving yields $N = 38$, hence the answer is **1938**.

Alternate Solution: Let N be the number formed by the rightmost two digits of the year in which Charlie was born. The number of years from 1900 to 2014 can be thought of as the number of years before Charlie was born plus the number of years since he was born, or N plus Charlie's age. Thus $N + 2N = 114$, which leads to $N = 38$, so the answer is **1938**.

Problem 2. Let A, B, and C be randomly chosen (not necessarily distinct) integers between 0 and 4 inclusive. Pat and Chris compute the value of $A + B \cdot C$ by two different methods. Pat follows the proper order of operations, computing $A+(B\cdot C)$. Chris ignores order of operations, choosing instead to compute $(A + B) \cdot C$. Compute the probability that Pat and Chris get the same answer.

Solution 2. If Pat and Chris get the same answer, then $A+(B\cdot C) = (A+B)\cdot C$, or $A+BC = AC+BC$, or $A = AC$. This equation is true if $A = 0$ or $C = 1$; the equation places no restrictions on B. There are 25 triples (A, B, C) where $A = 0$, 25 triples where $C = 1$, and 5 triples where $A = 0$ and $C = 1$. As all triples are equally likely, the answer is $\frac{25+25-5}{5^3} = \frac{45}{125} = \mathbf{\frac{9}{25}}$.

Problem 3. Bobby, Peter, Greg, Cindy, Jan, and Marcia line up for ice cream. In an *acceptable lineup*, Greg is ahead of Peter, Peter is ahead of Bobby, Marcia is ahead of Jan, and Jan is ahead of Cindy. For example, the lineup with Greg in front, followed by Peter, Marcia, Jan, Cindy, and Bobby, in that order, is an acceptable lineup. Compute the number of acceptable lineups.

Solution 3. There are 6 people, so there are $6! = 720$ permutations. However, for each arrangement of the boys, there are $3! = 6$ permutations of the girls, of which only one yields an acceptable lineup. The same logic holds for the boys. Thus the total number of permutations must be divided by $3! \cdot 3! = 36$, yielding $6!/(3! \cdot 3!) = \mathbf{20}$ acceptable lineups.

Alternate Solution: Once the positions of Greg, Peter, and Bobby are determined, the entire lineup is determined, because there is only one acceptable ordering of the three girls. Because the boys occupy three of the six positions, there are $\binom{6}{3} = \mathbf{20}$ acceptable lineups.

Problem 4. In triangle ABC, $a = 12$, $b = 17$, and $c = 13$. Compute $b \cos C - c \cos B$.

Solution 4. Using the Law of Cosines, $a^2 + b^2 - 2ab \cos C = c^2$ implies

$$b \cos C = \frac{a^2 + b^2 - c^2}{2a}.$$

Similarly,

$$c\cos B = \frac{a^2 - b^2 + c^2}{2a}.$$

Thus

$$\begin{aligned} b\cos C - c\cos B &= \frac{a^2 + b^2 - c^2}{2a} - \frac{a^2 - b^2 + c^2}{2a} \\ &= \frac{2b^2 - 2c^2}{2a} \\ &= \frac{b^2 - c^2}{a}. \end{aligned}$$

With the given values, the result is $(17^2 - 13^2)/12 = 120/12 = \mathbf{10}$.

Alternate Solution: Let H be the foot of the altitude from A to $\overline{BC}$; let $BH = x$, $CH = y$, and $AH = h$. Then $b\cos C = y$, $c\cos B = x$, and the desired quantity is $Q = y - x$. However, $y + x = a$, so $y^2 - x^2 = aQ$. By the Pythagorean Theorem, $y^2 = b^2 - h^2$ and $x^2 = c^2 - h^2$, so $y^2 - x^2 = (b^2 - h^2) - (c^2 - h^2) = b^2 - c^2$. Thus $aQ = b^2 - c^2$, and $Q = \frac{b^2 - c^2}{a}$ as in the first solution.

Problem 5. The sequence of words $\{a_n\}$ is defined as follows: $a_1 = X$, $a_2 = O$, and for $n \geq 3$, a_n is a_{n-1} followed by the reverse of a_{n-2}. For example, $a_3 = OX$, $a_4 = OXO$, $a_5 = OXOXO$, and $a_6 = OXOXOOXO$. Compute the number of palindromes in the first 1000 terms of this sequence.

Solution 5. Let P denote a palindromic word, let Q denote any word, and let $\overline{R}$ denote the reverse of word R. Note that if two consecutive terms of the sequence are $a_n = P$, $a_{n+1} = Q$, then $a_{n+2} = Q\overline{P} = QP$ and $a_{n+3} = QP\overline{Q}$. Thus if a_n is a palindrome, so is a_{n+3}. Because a_1 and a_2 are both palindromes, then so must be all terms in the subsequences $a_4, a_7, a_{10}, \ldots$ and $a_5, a_8, a_{11}, \ldots$.

To show that the other terms are not palindromes, note that if P' is *not* a palindrome, then $QP'\overline{Q}$ is also not a palindrome. Thus if a_n is not a palindrome, then a_{n+3} is not a palindrome either. Because $a_3 = OX$ is not a palindrome, neither is any term of the subsequence $a_6, a_9, a_{12}, \ldots$. (Alternatively, counting the number of X's in each word a_i shows that the number of X's in a_{3k} is odd. So if a_{3k} were to be a palindrome, it would have to have an odd number of letters, with an X in the middle. However, it can be shown that the length of a_{3k} is even. Thus a_{3k} cannot be a palindrome.)

In total there are $1000 - 333 = \mathbf{667}$ palindromes among the first 1000 terms.

Problem 6. Compute the smallest positive integer n such that $214 \cdot n$ and $2014 \cdot n$ have the same number of divisors.

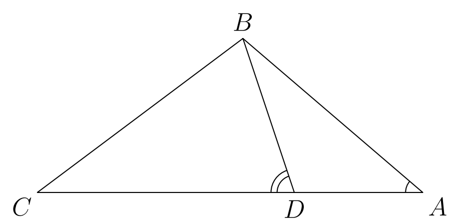
Solution 6. Let $D(n)$ be the number of divisors of the integer n. Note that if $D(214n) = D(2014n)$ and if some p divides n and is relatively prime to both 214 and 2014, then $D(\frac{214n}{p}) = D(\frac{2014n}{p})$. Thus any prime divisor of the smallest possible positive n will be a divisor of $214 = 2 \cdot 107$ or $2014 = 2 \cdot 19 \cdot 53$. For the sake of convenience, write $n = 2^{a-1}19^{b-1}53^{c-1}107^{d-1}$, where $a, b, c, d \geq 1$. Then $D(214n) = (a+1)bc(d+1)$ and $D(2014n) = (a+1)(b+1)(c+1)d$. Divide both sides by $a+1$ and expand to get $bcd + bc = bcd + bd + cd + d$, or $bc - bd - cd - d = 0$.

Because the goal is to minimize n, try $d = 1$: $bc - b - c - 1 = 0 \Rightarrow (b-1)(c-1) = 2$, which has solutions $(b, c) = (2, 3)$ and $(3, 2)$. The latter gives the smaller value for n, namely $19^2 \cdot 53 = 19133$. The only quadruples (a, b, c, d) that satisfy $2^{a-1}19^{b-1}53^{c-1}107^{d-1} < 19133$ and $d > 1$ are $(1, 1, 2, 2), (1, 2, 1, 2)$, and $(1, 1, 1, 3)$. None of these quadruples satisfies $bc - bd - cd - d = 0$, so the minimum value is $n = \mathbf{19133}$.

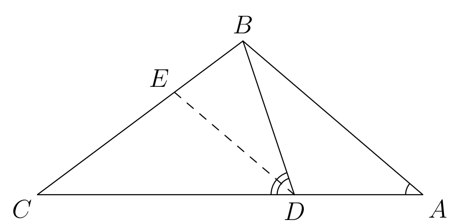
Problem 7. Let N be the least integer greater than 20 that is a palindrome in both base 20 and base 14. For example, the three-digit base-14 numeral $(13)5(13)_{14}$ (representing $13 \cdot 14^2 + 5 \cdot 14^1 + 13 \cdot 14^0$) is a palindrome in base 14, but not in base 20, and the three-digit base-14 numeral $(13)31_{14}$ is not a palindrome in base 14. Compute the base-10 representation of N.

Solution 7. Because N is greater than 20, the base-20 and base-14 representations of N must be at least two digits long. The smallest possible case is that N is a two-digit palindrome in both bases. Then $N = 20a + a = 21a$, where $1 \le a \le 19$. Similarly, in order to be a two-digit palindrome in base 14, $N = 14b + b = 15b$, with $1 \le b \le 13$. So N would have to be a multiple of both 21 and 15. The least common multiple of 21 and 15 is 105, which has the base 20 representation of $105 = 55_{20}$ and the base-14 representation of $105 = 77_{14}$, both of which are palindromes. Thus the answer is $\mathbf{105}$.

Problem 8. In triangle ABC, $BC = 2$. Point D is on $\overline{AC}$ such that $AD = 1$ and $CD = 2$. If $\text{m}\angle BDC = 2\text{m}\angle A$, compute $\sin A$.



Solution 8. Let $[ABC] = K$. Then $[BCD] = \frac{2}{3} \cdot K$. Let $\overline{DE}$ be the bisector of $\angle BDC$, as shown below.



Notice that $\text{m}\angle DBA = \text{m}\angle BDC - \text{m}\angle A = \text{m}\angle A$, so triangle ADB is isosceles, and $BD = 1$. (Alternately, notice that $\overline{DE} \parallel \overline{AB}$, and by similar triangles, $[CDE] = \frac{4}{9} \cdot K$, which means $[BDE] = \frac{2}{9} \cdot K$. Because $[CDE] : [BDE] = 2$ and $\angle BDE \cong \angle CDE$, conclude that $\frac{CD}{BD} = 2$, thus $BD = 1$.) Because BCD is isosceles, it follows that $\cos \angle BDC = \frac{1}{2}BD/CD = \frac{1}{4}$. By the half-angle formula,

$$\sin A = \sqrt{\frac{1 - \cos \angle BDC}{2}} = \sqrt{\frac{3}{8}} = \frac{\boldsymbol{\sqrt{6}}}{\mathbf{4}}.$$

Problem 9. Compute the greatest integer $k \leq 1000$ such that $\binom{1000}{k}$ is a multiple of 7.

Solution 9. The ratio of binomial coefficients $\binom{1000}{k}/\binom{1000}{k+1} = \frac{k+1}{1000-k}$. Because 1000 is 1 less than a multiple of 7, namely $1001 = 7 \cdot 11 \cdot 13$, either $1000 - k$ and $k + 1$ are both multiples of 7 or neither is. Hence whenever the numerator is divisible by 7, the denominator is also. Thus for the largest value of k such that $\binom{1000}{k}$ is a multiple of 7, $\frac{k+1}{1000-k}$ must equal $7 \cdot \frac{p}{q}$, where p and q are relatively prime integers and $7 \nmid q$. The only way this can happen is when $k + 1$ is a multiple of 49, the greatest of which less than 1000 is 980. Therefore the greatest value of k satisfying the given conditions is $980 - 1 = \mathbf{979}$.

Alternate Solution: Rewrite 1000 in base 7: $1000 = 2626_7$. Let $k = \underline{a}\,\underline{b}\,\underline{c}\,\underline{d}_7$. By Lucas's Theorem, $\binom{1000}{k} \equiv \binom{2}{a}\binom{6}{b}\binom{2}{c}\binom{6}{d} \bmod 7$. The binomial coefficient $\binom{p}{q} = 0$ only when $q > p$. Base 7 digits cannot exceed 6, and $k \leq 1000$, thus the greatest value of k that works is $2566_7 = \mathbf{979}$. (Alternatively, the least value of k that works is $30_7 = 21$; because $\binom{n}{k} = \binom{n}{n-k}$, the greatest such k is $1000 - 21 = 979$.)

Problem 10. An integer-valued function f is called *tenuous* if $f(x) + f(y) > x^2$ for all positive integers x and y. Let g be a tenuous function such that $g(1) + g(2) + \cdots + g(20)$ is as small as possible. Compute the minimum possible value for $g(14)$.

Solution 10. For a tenuous function g, let $S_g = g(1) + g(2) + \cdots + g(20)$. Then:

$$\begin{aligned} S_g &= (g(1) + g(20)) + (g(2) + g(19)) + \cdots + (g(10) + g(11)) \\ &\geq (20^2 + 1) + (19^2 + 1) + \cdots + (11^2 + 1) \\ &= 10 + \sum_{k=11}^{20} k^2 \\ &= 2495. \end{aligned}$$

The following argument shows that if a tenuous function g attains this sum, then $g(1) = g(2) = \cdots = g(10)$. First, if the sum equals 2495, then $g(1) + g(20) = 20^2 + 1, g(2) + g(19) = 19^2 + 1, \ldots, g(10) + g(11) = 11^2 + 1$. If $g(1) < g(2)$, then $g(1) + g(19) < 19^2 + 1$, which contradicts the tenuousness of g. Similarly, if $g(2) > g(1)$, then $g(2) + g(20) < 20^2 + 1$. Therefore $g(1) = g(2)$. Analogously, comparing $g(1)$ and $g(3)$, $g(1)$ and $g(4)$, etc. shows that $g(1) = g(2) = g(3) = \cdots = g(10)$.

Now consider all functions g for which $g(1) = g(2) = \cdots = g(10) = a$ for some integer a. Then $g(n) = n^2 + 1 - a$ for $n \geq 11$. Because $g(11) + g(11) > 11^2 = 121$, it is the case that $g(11) \geq 61$. Thus $11^2 + 1 - a \geq 61 \Rightarrow a \leq 61$. Thus the smallest possible value for $g(14)$ is $14^2 + 1 - 61 = \mathbf{136}$.

9 Relay Problems

Relay 1-1 Let $T = (0, 0)$, $N = (2, 0)$, $Y = (6, 6)$, $W = (2, 6)$, and $R = (0, 2)$. Compute the area of pentagon $TNYWR$.

Relay 1-2 Let $T = TNYWR$. The lengths of the sides of a rectangle are the zeroes of the polynomial $x^2 - 3Tx + T^2$. Compute the length of the rectangle's diagonal.

Relay 1-3 Let $T = TNYWR$. Let $w > 0$ be a real number such that T is the area of the region above the x-axis, below the graph of $y = \lceil x \rceil^2$, and between the lines $x = 0$ and $x = w$. Compute $\lceil 2w \rceil$.

Relay 2-1 Compute the least positive integer n such that $\gcd(n^3, n!) \geq 100$.

Relay 2-2 Let $T = TNYWR$. At a party, everyone shakes hands with everyone else exactly once, except Ed, who leaves early. A grand total of $20T$ handshakes take place. Compute the number of people at the party who shook hands with Ed.

Relay 2-3 Let $T = TNYWR$. Given the sequence u_n such that $u_3 = 5$, $u_6 = 89$, and $u_{n+2} = 3u_{n+1} - u_n$ for integers $n \geq 1$, compute u_T.

10 Relay Answers

Answer 1-1 20

Answer 1-2 $20\sqrt{7}$

Answer 1-3 10

Answer 2-1 8

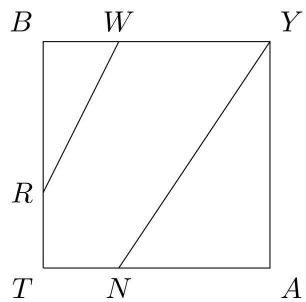
Answer 2-2 7

Answer 2-3 233

11 Relay Solutions

Relay 1-1 Let $T = (0,0)$, $N = (2,0)$, $Y = (6,6)$, $W = (2,6)$, and $R = (0,2)$. Compute the area of pentagon $TNYWR$.

Solution 1-1 Pentagon $TNYWR$ fits inside square $TAYB$, where $A = (6,0)$ and $B = (0,6)$. The region of $TAYB$ not in $TNYWR$ consists of triangles $\triangle NAY$ and $\triangle WBR$, as shown below.



Thus

$$\begin{aligned}[TNYWR] &= [TAYB] - [NAY] - [WBR] \\ &= 6^2 - \frac{1}{2} \cdot 4 \cdot 6 - \frac{1}{2} \cdot 2 \cdot 4 \\ &= \mathbf{20}.\end{aligned}$$

Relay 1-2 Let $T = TNYWR$. The lengths of the sides of a rectangle are the zeroes of the polynomial $x^2 - 3Tx + T^2$. Compute the length of the rectangle's diagonal.

Solution 1-2 Let r and s denote the zeros of the polynomial $x^2 - 3Tx + T^2$. The rectangle's diagonal has length $\sqrt{r^2 + s^2} = \sqrt{(r+s)^2 - 2rs}$. Recall that for a quadratic polynomial $ax^2 + bx + c$, the sum of its zeros is $-b/a$, and the product of its zeros is c/a. In this particular instance, $r + s = 3T$ and $rs = T^2$. Thus the length of the rectangle's diagonal is $\sqrt{9T^2 - 2T^2} = T \cdot \sqrt{7}$. With $T = 20$, the rectangle's diagonal is $\mathbf{20\sqrt{7}}$.

Relay 1-3 Let $T = TNYWR$. Let $w > 0$ be a real number such that T is the area of the region above the x-axis, below the graph of $y = \lceil x \rceil^2$, and between the lines $x = 0$ and $x = w$. Compute $\lceil 2w \rceil$.

Solution 1-3 Write $w = k + \alpha$, where k is an integer, and $0 \leq \alpha < 1$. Then

$$T = 1^2 + 2^2 + \cdots + k^2 + (k+1)^2 \cdot \alpha.$$

Computing $\lceil 2w \rceil$ requires computing w to the nearest half-integer. First obtain the integer k. As $\sqrt{7} > 2$, with $T = 20\sqrt{7}$, one obtains $T > 40$. As $1^2 + 2^2 + 3^2 + 4^2 = 30$, it follows that $k \geq 4$. To obtain an upper bound for k, note that $700 < 729$, so $10\sqrt{7} < 27$, and $T = 20\sqrt{7} < 54$. As $1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55$, it follows that $4 < w < 5$, and hence $k = 4$.

It now suffices to determine whether or not $\alpha > 0.5$. To this end, one must determine whether $T > 1^2 + 2^2 + 3^2 + 4^2 + 5^2/2 = 42.5$. Indeed, note that $2.5^2 = 6.25 < 7$, so $T > (20)(2.5) = 50$. It follows that $\alpha > 0.5$, so $4.5 < w < 5$. Thus $9 < 2w < 10$, and $\lceil 2w \rceil = \mathbf{10}$.

Alternate Solution: Once it has been determined that $4 < w < 5$, the formula for T yields $1 + 4 + 9 + 16 + 25 \cdot \alpha = 20\sqrt{7}$, hence $\alpha = \frac{4\sqrt{7}-6}{5}$. Thus $2\alpha = \frac{8\sqrt{7}-12}{5} = \frac{\sqrt{448}-12}{5} > \frac{21-12}{5} = 1.8$. Because $2w = 2k + 2\alpha$, it follows that $\lceil 2w \rceil = \lceil 8 + 2\alpha \rceil = \mathbf{10}$, because $1.8 < 2\alpha < 2$.

Relay 2-1 Compute the least positive integer n such that $\gcd(n^3, n!) \geq 100$.

Solution 2-1 Note that if p is prime, then $\gcd(p^3, p!) = p$. A good strategy is to look for values of n with several (not necessarily distinct) prime factors so that n^3 and $n!$ will have many factors in common. For example, if $n = 6$, $n^3 = 216 = 2^3 \cdot 3^3$ and $n! = 720 = 2^4 \cdot 3^2 \cdot 5$, so $\gcd(216, 720) = 2^3 \cdot 3^2 = 72$. Because 7 is prime, try $n = 8$. Notice that $8^3 = 2^9$ while $8! = 2^7 \cdot 3^2 \cdot 5 \cdot 7$. Thus $\gcd(512, 8!) = 2^7 = 128 > 100$, hence the smallest value of n is **8**.

Relay 2-2 Let $T = TNYWR$. At a party, everyone shakes hands with everyone else exactly once, except Ed, who leaves early. A grand total of $20T$ handshakes take place. Compute the number of people at the party who shook hands with Ed.

Solution 2-2 If there were n people at the party, including Ed, and if Ed had not left early, there would have been $\binom{n}{2}$ handshakes. Because Ed left early, the number of handshakes is strictly less than that, but greater than $\binom{n-1}{2}$ (everyone besides Ed shook everyone else's hand). So find the least number n such that $\binom{n}{2} \geq 160$. The least such n is 19, because $\binom{18}{2} = 153$ and $\binom{19}{2} = 171$. Therefore there were 19 people at the party. However, $171 - 160 = 11$ handshakes never took place. Therefore the number of people who shook hands with Ed is $19 - 11 - 1 = \mathbf{7}$.

Relay 2-3 Let $T = TNYWR$. Given the sequence u_n such that $u_3 = 5$, $u_6 = 89$, and $u_{n+2} = 3u_{n+1} - u_n$ for integers $n \geq 1$, compute u_T.

Solution 2-3 By the recursive definition, notice that $u_6 = 89 = 3u_5 - u_4$ and $u_5 = 3u_4 - u_3 = 3u_4 - 5$. This is a linear system of equations. Write $3u_5 - u_4 = 89$ and $-3u_5 + 9u_4 = 15$ and add to obtain $u_4 = 13$. Now apply the recursive definition to obtain $u_5 = 34$ and $u_7 = \mathbf{233}$.

Alternate Solution: Notice that the given values are both Fibonacci numbers, and that in the Fibonacci sequence, $f_1 = f_2 = 1$, $f_5 = 5$, and $f_{11} = 89$. That is, 5 and 89 are six terms apart in the Fibonacci sequence, and only three terms apart in the given sequence. This relationship is not a coincidence: alternating terms in the Fibonacci sequence satisfy the given recurrence relation for the sequence $\{u_n\}$, that is, $f_{n+4} = 3f_{n+2} - f_n$. Proof: if $f_n = a$ and $f_{n+1} = b$, then $f_{n+2} = a + b$, $f_{n+3} = a + 2b$, and $f_{n+4} = 2a + 3b = 3(a + b) - b = 3f_{n+2} - f_n$. To compute the final result, continue out the Fibonacci sequence to obtain $f_{12} = 144$ and $u_7 = f_{13} = \mathbf{233}$.

12 Super Relay

1. The sequence $a_1, a_2, a_3, \ldots$ is a geometric sequence with $a_{20} = 8$ and $a_{14} = 2^{21}$. Compute a_{21}.

2. Let $T = TNYWR$. Circles L and O are internally tangent and have radii T and $4T$, respectively. Point E lies on circle L such that $\overline{OE}$ is tangent to circle L. Compute OE.

3. Let $T = TNYWR$. In a right triangle, one leg has length T^2 and the other leg is 2 less than the hypotenuse. Compute the triangle's perimeter.

4. Let $T = TNYWR$. If $x + 9y = 17$ and $Tx + (T+1)y = T + 2$, compute $20x + 14y$.

5. Let $T = TNYWR$. Let $f(x) = ax^2 + bx + c$. The product of the roots of f is T. If $(-2, 20)$ and $(1, 14)$ lie on the graph of f, compute a.

6. Let $T = TNYWR$. Let $z_1 = 15 + 5i$ and $z_2 = 1 + Ki$. Compute the smallest positive integral value of K such that $|z_1 - z_2| \geq 15T$.

7. Let $T = TNYWR$. Suppose that T people are standing in a line, including three people named Charlie, Chris, and Abby. If the people are assigned their positions in line at random, compute the probability that Charlie is standing next to at least one of Chris or Abby.

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15. Compute the smallest positive integer N such that $20N$ is a multiple of 14 and $14N$ is a multiple of 20.

14. Let $T = TNYWR$. Chef Selma is preparing a burrito menu. A burrito consists of: (1) a choice of chicken, beef, turkey, or no meat, (2) exactly one of three types of beans, (3) exactly one of two types of rice, and (4) exactly one of K types of cheese. Compute the smallest value of K such that Chef Selma can make at least T different burrito varieties.

13. Let $T = TNYWR$. Regular hexagon $SUPERB$ has side length $\sqrt{T}$. Compute the value of $BE \cdot SU \cdot RE$.

12. Let $T = TNYWR$. Compute $\sqrt{\sqrt{\sqrt[T]{10^{T^2-T}}}}$.

11. Let $T = TNYWR$. Nellie has a flight from Rome to Athens that is scheduled to last for $T + 30$ minutes. However, owing to a tailwind, her flight only lasts for T minutes. The plane's speed is 1.5 miles per minute faster than what it would have been for the originally scheduled flight. Compute the distance (in miles) that the plane travels.

10. Let $T = TNYWR$. If $\log T = 2 - \log 2 + \log k$, compute the value of k.

9. Let $T = TNYWR$. If r is the radius of a right circular cone and the cone's height is $T - r^2$, let V be the maximum possible volume of the cone. Compute π / V.

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8. Let A be the number you will receive from position 7 and let B be the number you will receive from position 9. Let $\alpha = \sin^{-1} A$ and let $\beta = \cos^{-1} B$. Compute $\sin(\alpha + \beta) + \sin(\alpha - \beta)$.

13 Super Relay Answers

1. 1

2. $2\sqrt{2}$

3. 40

4. 8

5. 8/5

6. 25

7. 47/300

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15. 70

14. 3

13. 9

12. 100

11. 650

10. 13

9. 12/169

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8. 94/4225

14 Super Relay Solutions

Problem 1. The sequence $a_1, a_2, a_3, \ldots$ is a geometric sequence with $a_{20} = 8$ and $a_{14} = 2^{21}$. Compute a_{21}.

Solution 1. Let r be the common ratio of the sequence. Then $a_{20} = r^{20-14} \cdot a_{14}$, hence $8 = r^6 \cdot 2^{21} \Rightarrow r^6 = \frac{2^3}{2^{21}} = 2^{-18}$, so $r = 2^{-3} = \frac{1}{8}$. Thus $a_{21} = r \cdot a_{20} = \frac{1}{8} \cdot 8 = \mathbf{1}$.

Problem 2. Let $T = TNYWR$. Circles L and O are internally tangent and have radii T and $4T$, respectively. Point E lies on circle L such that $\overline{OE}$ is tangent to circle L. Compute OE.

Solution 2. Because $\overline{OE}$ is tangent to circle L, $\overline{LE} \perp \overline{OE}$. Also note that $LO = 4T - T = 3T$. Hence, by the Pythagorean Theorem, $OE = \sqrt{(3T)^2 - T^2} = 2T\sqrt{2}$ (this also follows from the Tangent-Secant Theorem). With $T = 1$, $OE = \mathbf{2\sqrt{2}}$.

Problem 3. Let $T = TNYWR$. In a right triangle, one leg has length T^2 and the other leg is 2 less than the hypotenuse. Compute the triangle's perimeter.

Solution 3. Let c be the length of the hypotenuse. Then, by the Pythagorean Theorem, $(T^2)^2 + (c-2)^2 = c^2 \Rightarrow c = \frac{T^4}{4} + 1$. With $T = 2\sqrt{2}$, $T^4 = 64$, and $c = 17$. So the triangle is a $8-15-17$ triangle with perimeter **40**.

Problem 4. Let $T = TNYWR$. If $x + 9y = 17$ and $Tx + (T+1)y = T + 2$, compute $20x + 14y$.

Solution 4. Multiply each side of the first equation by T to obtain $Tx + 9Ty = 17T$. Subtract the second equation to yield $9Ty - Ty - y = 16T - 2 \Rightarrow y(8T-1) = 2(8T-1)$. Hence either $T = \frac{1}{8}$ (in which case, the value of y is not uniquely determined) or $y = 2$. Plug $y = 2$ into the first equation to obtain $x = -1$. Hence $20x + 14y = -20 + 28 = \mathbf{8}$.

Problem 5. Let $T = TNYWR$. Let $f(x) = ax^2 + bx + c$. The product of the roots of f is T. If $(-2, 20)$ and $(1, 14)$ lie on the graph of f, compute a.

Solution 5. Using Vièta's Formula, write $f(x) = ax^2 + bx + Ta$. Substituting the coordinates of the given points yields the system of equations: $4a - 2b + Ta = 20$ and $a + b + Ta = 14$. Multiply each side of the latter equation by 2 and add the resulting equation to the former equation to eliminate b. Simplifying yields $a = \frac{16}{T+2}$. With $T = 8$, $a = \mathbf{8/5}$.

Problem 6. Let $T = TNYWR$. Let $z_1 = 15 + 5i$ and $z_2 = 1 + Ki$. Compute the smallest positive integral value of K such that $|z_1 - z_2| \geq 15T$.

Solution 6. Note that $z_1 - z_2 = 14 + (5-K)i$, hence $|z_1 - z_2| = \sqrt{14^2 + (5-K)^2}$. With $T = 8/5, 15T = 24$, hence $14^2 + (5-K)^2 \geq 24^2$. Thus $|5 - K| \geq \sqrt{24^2 - 14^2} = \sqrt{380}$. Because K is a positive integer, it follows that $K - 5 \geq 20$, hence the desired value of K is **25**.

Problem 7. Let $T = TNYWR$. Suppose that T people are standing in a line, including three people named Charlie, Chris, and Abby. If the people are assigned their positions in line at random, compute the probability that Charlie is standing next to at least one of Chris or Abby.

Solution 7. First count the number of arrangements in which Chris stands next to Charlie. This is $(T-1) \cdot 2! \cdot (T-2)! = 2 \cdot (T-1)!$ because there are $T - 1$ possible leftmost positions for the pair $\{\text{Charlie}, \text{Chris}\}$, there are $2!$ orderings of this pair, and there are $(T-2)!$ ways to arrange the remaining people. There are equally many arrangements in which Abby stands next to Charlie. However, adding these overcounts the arrangements in which Abby, Charlie, and Chris are standing next to each other, with Charlie in the middle. Using similar reasoning as above, there are $(T-2)\cdot 2!\cdot(T-3)! = 2\cdot(T-2)!$ such arrangements. Hence the desired probability is $\frac{2\cdot 2\cdot(T-1)!-2\cdot(T-2)!}{T!} = \frac{2\cdot(T-2)!(2T-2-1)}{T!} = \frac{2(2T-3)}{T(T-1)}$. With $T=25$, the fraction simplifies to $\mathbf{\frac{47}{300}}$.

Problem 15. Compute the smallest positive integer N such that $20N$ is a multiple of 14 and $14N$ is a multiple of 20.

Solution 15. Because $\gcd(14,20)=2$, the problem is equivalent to computing the smallest positive integer N such that $7 \mid 10N$ and $10 \mid 7N$. Thus $7 \mid N$ and $10 \mid N$, and the desired value of N is $\operatorname{lcm}(7,10) = \mathbf{70}$.

Problem 14. Let $T = TNYWR$. Chef Selma is preparing a burrito menu. A burrito consists of: (1) a choice of chicken, beef, turkey, or no meat, (2) exactly one of three types of beans, (3) exactly one of two types of rice, and (4) exactly one of K types of cheese. Compute the smallest value of K such that Chef Selma can make at least T different burrito varieties.

Solution 14. Using the Multiplication Principle, Chef Selma can make $4\cdot 3\cdot 2\cdot K = 24K$ different burrito varieties. With $T=70$, the smallest integral value of K such that $24K \geq 70$ is $\left\lceil \frac{70}{24} \right\rceil = \mathbf{3}$.

Problem 13. Let $T = TNYWR$. Regular hexagon $SUPERB$ has side length $\sqrt{T}$. Compute the value of $BE \cdot SU \cdot RE$.

Solution 13. Because $\overline{SU}$ and $\overline{RE}$ are sides of the hexagon, $SU = RE = \sqrt{T}$. Let H be the foot of the altitude from R to $\overline{BE}$ in $\triangle BRE$ and note that each interior angle of a regular hexagon is 120°. Thus $BE = BH + HE = 2\left(\frac{\sqrt{3}}{2}\right)\left(\sqrt{T}\right) = \sqrt{3T}$. Thus $BE\cdot SU\cdot RE = \sqrt{3T}\cdot\sqrt{T}\cdot\sqrt{T} = T\sqrt{3T}$. With $T=3$, the answer is $\mathbf{9}$.

Problem 12. Let $T = TNYWR$. Compute $\sqrt{\sqrt{\sqrt[T]{10^{T^2-T}}}}$.

Solution 12. The given radical equals $\left(\left(\left(10^{T^2-T}\right)^{\frac{1}{T}}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}} = 10^{(T-1)/4}$. With $T=9$, this simplifies to $10^2 = \mathbf{100}$.

Problem 11. Let $T = TNYWR$. Nellie has a flight from Rome to Athens that is scheduled to last for $T+30$ minutes. However, owing to a tailwind, her flight only lasts for T minutes. The plane's speed is 1.5 miles <u>per minute</u> faster than what it would have been for the originally scheduled flight. Compute the distance (in miles) that the plane travels.

Solution 11. Let D be the distance in miles traveled by the plane. The given conditions imply that $\frac{D}{T} - \frac{D}{T+30} = 1.5 \Rightarrow \frac{30D}{T(T+30)} = 1.5 \Rightarrow D = \frac{T(T+30)}{20}$. With $T=100, D = 5\cdot 130 = \mathbf{650}$.

Problem 10. Let $T = TNYWR$. If $\log T = 2 - \log 2 + \log k$, compute the value of k.

Solution 10. Write $2 = \log 100$ and use the well-known properties for the sum/difference of two logs to obtain $\log T = \log\left(\frac{100k}{2}\right)$, hence $k = \frac{T}{50}$. With $T=650$, $k = \mathbf{13}$.

Problem 9. Let $T = TNYWR$. If r is the radius of a right circular cone and the cone's height is $T - r^2$, let V be the maximum possible volume of the cone. Compute π/V.

Solution 9. The cone's volume is $\frac{1}{3}\pi r^2(T - r^2)$. Maximizing this is equivalent to maximizing $x(T - x)$, where $x = r^2$. Using the formula for the vertex of a parabola (or the AM-GM inequality), the maximum value occurs when $x = \frac{T}{2}$. Hence $V = \frac{1}{3}\pi \cdot \frac{T}{2} \cdot \frac{T}{2} = \frac{\pi T^2}{12}$, and $\pi/V = 12/T^2$. With $T = 13$, $V = \mathbf{\frac{12}{169}}$.

Problem 8. Let A be the number you will receive from position 7 and let B be the number you will receive from position 9. Let $\alpha = \sin^{-1} A$ and let $\beta = \cos^{-1} B$. Compute $\sin(\alpha + \beta) + \sin(\alpha - \beta)$.

Solution 8. The given conditions are equivalent to $\sin\alpha = A$ and $\cos\beta = B$. Using either the sum-to-product or the sine of a sum/difference identities, the desired expression is equivalent to $2(\sin\alpha)(\cos\beta) = 2 \cdot A \cdot B$. With $A = \frac{47}{300}$ and $B = \frac{12}{169}$, $2 \cdot A \cdot B = \frac{2 \cdot 47}{25 \cdot 169} = \mathbf{\frac{94}{4225}}$.

15 Tiebreaker Problems

Problem 1. A student computed the repeating decimal expansion of $\frac{1}{N}$ for some integer N, but inserted six extra digits into the repetend to get $.\overline{0231846597}$. Compute the value of N.

Problem 2. Let n be a four-digit number whose square root is three times the sum of the digits of n. Compute n.

Problem 3. Compute the sum of the reciprocals of the positive integer divisors of 24.

16 Tiebreaker Answers

Answer 1. 606

Answer 2. 2916

Answer 3. 5/2

17 Tiebreaker Solutions

Problem 1. A student computed the repeating decimal expansion of $\frac{1}{N}$ for some integer N, but inserted six extra digits into the repetend to get $.\overline{0023184659}7$. Compute the value of N.

Solution 1. Because the given repetend has ten digits, the original had four digits. If $\frac{1}{N} = .0\overline{\underline{A}\,\underline{B}\,\underline{C}\,\underline{D}} = \frac{\underline{A}\,\underline{B}\,\underline{C}\,\underline{D}}{99990}$, then the numerator must divide $99990 = 10 \cdot 99 \cdot 101 = 2 \cdot 3^2 \cdot 5 \cdot 11 \cdot 101$.

Note that all 3- and 4-digit multiples of 101 contain at least one digit which appears twice. Because the 10-digit string under the vinculum (i.e., 0231846597) contains no repeated digits, $\underline{A}\,\underline{B}\,\underline{C}\,\underline{D}$ cannot be a multiple of 101. So $\underline{A}\,\underline{B}\,\underline{C}\,\underline{D}$ divides $2 \cdot 3^2 \cdot 5 \cdot 11 = 990$. The only divisor of 990 that can be formed from four of the given digits (taken in order) is 0165, that is, 165. Hence $\frac{1}{N} = \frac{165}{99990} = \frac{1}{606} \Rightarrow N = \mathbf{606}$.

Problem 2. Let n be a four-digit number whose square root is three times the sum of the digits of n. Compute n.

Solution 2. Because $\sqrt{n}$ is a multiple of 3, n must be a multiple of 9. Therefore the sum of the digits of n is a multiple of 9. Thus $\sqrt{n}$ must be a multiple of 27, which implies that n is a multiple of 27^2. The only candidates to consider are 54^2 $(= 2916)$ and 81^2 $(= 6561)$, and only **2916** satisfies the desired conditions.

Problem 3. Compute the sum of the reciprocals of the positive integer divisors of 24.

Solution 3. The map $n \mapsto 24/n$ establishes a one-to-one correspondence among the positive integer divisors of 24. Thus

$$\sum_{\substack{n|24\\n>0}} \frac{1}{n} = \sum_{\substack{n|24\\n>0}} \frac{1}{24/n} = \frac{1}{24}\sum_{\substack{n|24\\n>0}} n.$$

Because $24 = 2^3 \cdot 3$, the sum of the positive divisors of 24 is $(1 + 2 + 2^2 + 2^3)(1 + 3) = 15 \cdot 4 = 60$. Hence the sum is $60/24 = \mathbf{5/2}$.

Alternate Solution: Because $24 = 2^3 \cdot 3$, any positive divisor of 24 is of the form $2^a 3^b$ where $a = 0, 1, 2$, or 3, and $b = 0$ or 1. So the sum of the positive divisors of 24 can be represented as the product $(1 + 2 + 4 + 8)(1 + 3)$. Similarly, the sum of their reciprocals can be represented as the product $\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right)\left(\frac{1}{1} + \frac{1}{3}\right)$. The first sum is $\frac{15}{8}$ and the second is $\frac{4}{3}$, so the product is $\mathbf{5/2}$.