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MOCK EXAMINATION

AMC 12

American Mathematics Contest 12

Test Sample

Detailed Solutions

Make time to take the practice test.

It's one of the best ways to get ready for the AMC.

AMC 12 Mock Test

Detailed Solutions

Problem 1

A bag contains 9 blue marbles, a number of green marbles, and no other marbles. If $5/6$ of the marbles in the bag are green, then what is the number of green marbles in the bag?

(A) 54 (B) 45 (C) 40 (D) 36 (E) 30

Answer: **(B)**

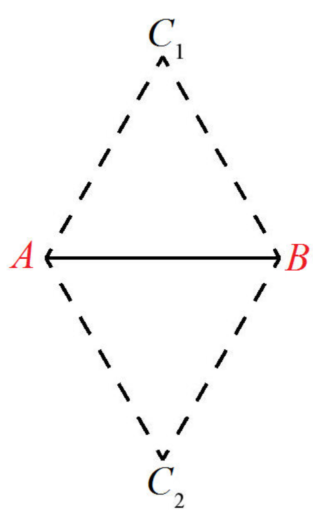
Since $5/6$ of the marbles are green and the remainder of the marbles are blue, it follows that the ratio of the green marbles to the blue marbles is $5 : 1$.

Note that the number of the blue marbles is 8. Hence, the number of the green marbles is:

$$8 \times 5 = 45.$$

Problem 2

Answer: **(B)**



If point C is placed so that such that

$$AB = BC = CA,$$

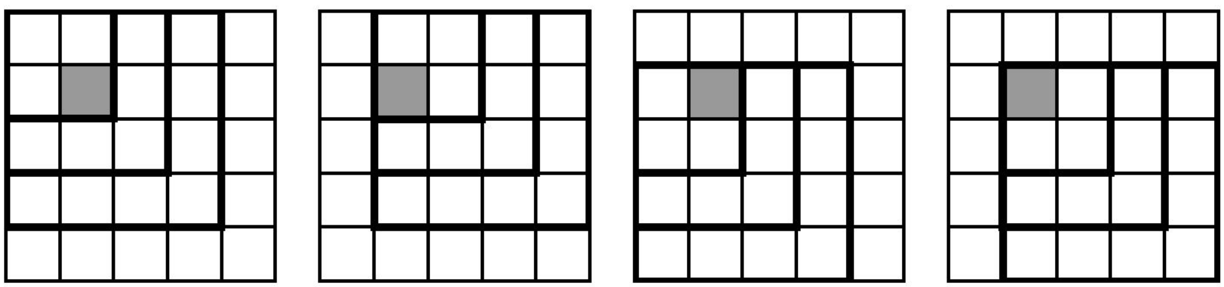
then the resulting ΔABC is equilateral. There are exactly two such possible equilateral triangles with base AB: one is with vertex C above AB, and the other is below AB.

Problem 3

Answer: (E)

There is 1 unit square that contains the shaded square (namely, the square itself).

There are 4 squares of each of the sizes $2 \times 2, 3 \times 3$, and 4×4 that contain the shaded square.



Finally, there is 1 square that is 5×5 that contains the shaded square (namely, the 5×5 grid itself).

In total, there are thus

$$1 + 4 + 4 + 4 + 1 = 14$$

squares that contain the shaded unit square.

Problem 4

Answer: (A)

Let us detect the first time after 4:56 where the digits are consecutive digits in increasing order. Note that 5:67 is not a valid time, and the time cannot start with 6, 7, 8, or 9. In addition, the digits of the time starting with 10 or 11 cannot be consecutive in increasing order.

Starting with 12, we get the time 12:34. This is the first one to fit the given conditions.

We have to calculate the length of time between 4:56 and 12:34.

From 4:56 to 12:56 is 8 hours, or

$$8 \times 60 = 480 \text{ minutes.}$$

From 12:34 to 12:56 is

$$56 - 34 = 22 \text{ minutes.}$$

Therefore, from 4:56 to 12:34 is

$$480 - 22 = 458 \text{ minutes.}$$

Problem 5

Answer: **(A)**

After translated 2 units to the left and 4 units up, the equation of the resulting line is

$$y - 4 = 2(x + 2),$$

or

$$y = 2x + 8.$$

Setting $y = 0$ to find the x-intercept yields:

$$x = -4.$$

Problem 6

Answer: **(D)**

Note that among 5 numbers, there are 3 powers of 2 (namely, 2, 4 and 8) and 2 integers that are not a power of 2 (namely, 6 and 10).

This means that the probability of choosing a power of 2 at random from the sets $\{2, 4, 6, 8, 10\}$ is

$$\frac{3}{5}.$$

Thus, the probability that the product of the numbers on the 3 dice is a power of 2 is:

$$\left(\frac{3}{5}\right)^3 = \frac{27}{125}.$$

By complementary probability, the probability that the product of the numbers on the 3 dice is not a power of 2 is:

$$1 - \frac{27}{125} = \frac{98}{125}.$$

Problem 7

Answer: **(C)**

Since $1 - i$ and i are roots of the real polynomial $P(x)$, it follows that their conjugates $1 + i$ and $-i$ are also roots of $P(x)$. Thus,

$$P(x) = \big(x - (1 - i)\big)\big(x - (1 + i)\big)(x - i)(x - i)$$
$$= ((x - 1)^2 + 1)(x^2 + 1) = (x^2 - 2x + 2)(x^2 + 1)$$
$$= x^4 - 2x^3 + 3x^2 - 2x + 2.$$

Plugging in $x = 1$ gives:

$$P(1) = 1 + a_1 + a_2 + a_3 + a_4 = 1^4 - 2 \cdot 1^3 + 3 \cdot 1^2 - 2 \cdot 1 + 2 = 2.$$

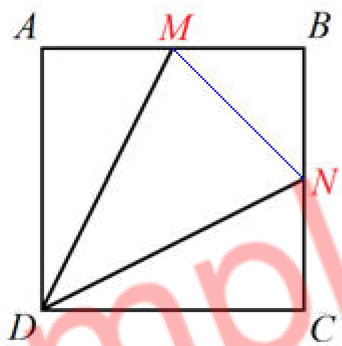
Hence,

$$a_1 + a_2 + a_3 + a_4 = 2 - 1 = 1.$$

Problem 8

Answer: **(B)**

Solution 1:



Note that

$$AD = 1, \qquad AM = \frac{1}{2}.$$

Using the Pythagorean Theorem in ΔADM gets

$$DM = \sqrt{AD^2 + AM^2} = \sqrt{1^2 + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{5}}{2}.$$

By symmetry,

$$DN = \frac{\sqrt{5}}{2}.$$

Since ΔMBN is an isosceles right triangle with leg length $\frac{1}{2}$, it follows that the hypotenuse is

$$MN = \frac{\sqrt{2}}{2}.$$

Applying the Law of Cosines to ΔDMN, we get

$$MN^2 = DM^2 + DM^2 - 2 \cdot DM \cdot DN \cdot \cos \angle MDN,$$

or

$$\left(\frac{\sqrt{2}}{2}\right)^2 = \left(\frac{\sqrt{5}}{2}\right)^2 + \left(\frac{\sqrt{5}}{2}\right)^2 - 2 \cdot \left(\frac{\sqrt{5}}{2}\right) \cdot \left(\frac{\sqrt{5}}{2}\right) \cdot \cos \angle MDN,$$

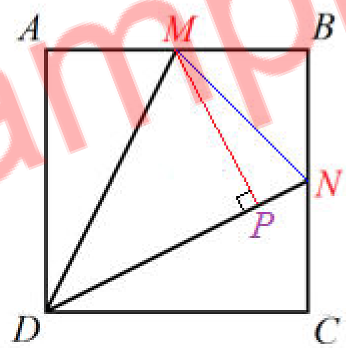
which implies that

$$\cos \angle MDN = \frac{4}{5}.$$

Hence,

$$\sin\angle MDN = \sqrt{1-\cos^2\angle MDN} = \sqrt{1-\left(\frac{4}{5}\right)^2} = \frac{3}{5}.$$

Solution 2:



Let P be the foot of the perpendicular from M to DN.

Place the figure in the coordinate plane with the origin at D, DC on the positive x-axis, and DA on the positive y-axis. Then $M = \left(\frac{1}{2}, 1\right)$ and $N = \left(1, \frac{1}{2}\right)$, so line DN has the equation

$$y = \frac{1}{2}x,$$

which is equivalent to

$$x - 2y = 0.$$

Using the point-to-line distance formula, we have:

$$MP = \frac{\left|\left(\frac{1}{2}\right) - 2(1)\right|}{\sqrt{1^2+(-2)^2}} = \frac{3}{2\sqrt{5}}.$$

Using the distance formula, we have:

$$DM = \sqrt{\left(\frac{1}{2}\right)^2 + 1^2} = \frac{\sqrt{5}}{2}.$$

Thus,

$$\sin \angle MDN = \frac{MP}{DM} = \frac{\frac{3}{2\sqrt{5}}}{\frac{\sqrt{5}}{2}} = \frac{3}{5}.$$

Problem 9

Answer: **(E)**

Solution 1:

Note that

$$\sqrt{48} = 4\sqrt{3}.$$

Thus,

$$7 + \sqrt{48} = 7 + 4\sqrt{3} = 2^2 + 2 \cdot 2 \cdot \sqrt{3} + \left(\sqrt{3}\right)^2 = \left(2 + \sqrt{3}\right)^2.$$

It follows that

$$\sqrt{7 + \sqrt{48}} = 2 + \sqrt{3}.$$

Hence,

$$a = 2, \qquad b = 3,$$

and

$$ab = 6.$$

Solution 2:

Squaring both sides gives:

$$\sqrt{7 + \sqrt{48}} = a + \sqrt{b}.$$

$$7 + \sqrt{48} = a^2 + b + 2a\sqrt{b}.$$

Suppose that

$$a^2 + b = 7 \text{ and } 2a\sqrt{b} = \sqrt{48}.$$

Squaring both sides of the second equation yields:

$$4a^2b = 48,$$

or

$$a^2b = 12.$$

Substituting $a^2 = 7 - b$ into the equation above gets:

$$(7-b)b = 12,$$

which is equivalent to

$$(b-3)(b-4) = 0.$$

Solving gives:

$$b = 3 \text{ or } 4.$$

If $b = 4$, then

$$a^2 = 7 - 4 = 3,$$

which is impossible because a is an integer.

If $b = 3$, then

$$a^2 = 7 - 3 = 4,$$

which implies the positive integer solution

$$a = 2.$$

Therefore, $a = 2$ and $b = 3$, which gives

$$ab = 6.$$

Problem 10

Answer: (B)

Since m and n are respectively the numbers of digits in 4^{2019} and 25^{2019}, it follows that

$$10^{m-1} < 4^{2019} < 10^m$$

and

$$10^{n-1} < 25^{2019} < 10^n.$$

Thus,

$$10^{m-1} \cdot 10^{n-1} < 4^{2019} \cdot 25^{2019} < 10^m \cdot 10^n,$$

or

$$10^{m+n-2} < 10^{4038} < 10^{m+n},$$

which implies that

$$4038 = m + n - 1.$$

Hence,

$$m + n = 4039.$$

Problem 11

Answer: **(D)**

Let r and h be the radius and the height of the cylinder, respectively.
According to the given conditions, we have:

$$\pi r^2 h = 10(2\pi r^2 + 2\pi r h),$$

which reduces to

$$rh = 20r + 20h.$$

This is equivalent to

$$(r - 20)(h - 20) = 400.$$

Let $r - 20 = a$ and $h - 20 = b$. Then

$$ab = 400.$$

Suppose $a < 0$. Then $b < 0$. If $a \le b$, then

$$a \le -20,$$

and

$$r = 20 + a \le 0,$$

which is impossible.
If $a \ge b$, then

$$b \le -20,$$

and

$$h = 20 + b \leq 0,$$

which is impossible.

So both $r - 20$ and $h - 20$ must be positive divisors of 400.

The number of positive divisors of $400 = 2^4 \cdot 5^2$ is

$$(4 + 1)(2 + 1) = 15.$$

Hence, there are 15 closed right cylinders to satisfy the given conditions.

Problem 12

Answer: (A)

The prime factorization of 6^{19} is

$$2^{19} \cdot 3^{19}.$$

So it has

$$(19 + 1)(19 + 1) = 400$$

positive integer divisors.

Note that the number of divisors of 6^{19} that are divisible by 6^{10} is equal to the number of divisors of

$$\frac{6^{19}}{6^{10}} = 6^9 = 2^9 \cdot 3^9,$$

which is

$$(9 + 1)(9 + 1) = 100.$$

Consequently, the desired probability is

$$\frac{100}{400} = \frac{1}{4}.$$

Problem 13

Answer: **(D)**

Solution 1

Note that at a time of 0 minutes, Alan was at the 31 meters mark.

Let Alan run x meters over these 3 minutes. Then he will be at the $x + 31$ meters mark after 3 minutes.

Since Ben is 20 meters ahead of Alan after 3 minutes, Ben is at the

$$x + 31 + 20 = x + 51$$

meters mark.

Since each runs at a constant speed, Ben runs

$$\frac{51}{3} = 17$$

meters per minute faster than Alan.

Since Ben finishes the race after 7 minutes, it follows that Ben runs for another 4 minutes.

Over these 4 minutes, he runs

$$4 \times 17 = 68$$

meters farther than Alan.

Recall that after the first 3 minutes, Ben was 20 meters ahead of Alan.

Hence, after 7 minutes, Ben was

$$20 + 68 = 88$$

meters farther ahead than Alan, and so Alan was 88 meters from the finish line.

Solution 2

Let Alan run x meters over the first 3 minutes. Then Ben ran

$$x + 31 + 20 = x + 51$$

meters over these 3 minutes.

Since Ben's speed is constant, he ran

$$\frac{4}{3}(x + 51)$$

meters over the next 4 minutes.

Since Alan's speed is constant, he ran $\frac{4}{3}x$ over these 4 minutes.

Thus, Ben ran a total of

$$(x+51)+\frac{4}{3}(x+51)=\frac{7}{3}x+119$$

meters.

Also, Alan was

$$31+x+\frac{4}{3}x=\frac{7}{3}x+31$$

meters far away from the starting line, because he had a 31 meters head start.

Hence, Alan's distance from the finish line, in meters, was

$$\left(\frac{7}{3}x+119\right)-\left(\frac{7}{3}x+31\right)=88.$$

Problem 14

Answer: (B)

Let

$$f(x)=(x-1)(x-3)(x-5)\cdots(x-2017)(x-2019).$$

Note that whenever an odd number of the 1010 factors of $f(x)$ are negative,

$$f(x)<0;$$

and

$$x-1>x-3>x-5>\cdots>x-2017>x-2019.$$

When $x=2$, we have $x-1=1$ and so all the other 1009 factors are negative, making

$$f(x)<0.$$

When $x=4$, we have $x-1=3$, $x-3=1$ and all of the other 1008 factors are negative, giving

$$f(x)>0.$$

When $x=6$, we have $x-1=5, x-3=3, x-5=1$ and all of the other 1007 factors are negative,

$$f(x)<0.$$

This pattern continues giving a negative value of $f(x)$ for

$$x = 2, 6, 10, 14, \cdots, 2014, 2018,$$

which an arithmetic sequence with the first term 1 and common difference 4. So there are

$$1 + \frac{2018 - 2}{4} = 505$$

such values.

When $x > 2019$, each factor is positive and so $f(x) > 0$.

Hence, there are 505 positive integers x for which

$$f(x) < 0.$$

Problem 15

Answer: **(A)**

Let

$$a = \sqrt[3]{5 + \sqrt{17}} \text{ and } b = \sqrt[3]{5 - \sqrt{17}}.$$

Then

$$x = a + b,$$

and

$$x^3 = (a + b)^3 = a^3 + b^3 + 3ab(a + b).$$

Note that

$$a^3 + b^3 = \left(5 + \sqrt{17}\right) + \left(5 - \sqrt{17}\right) = 10$$

and

$$ab = \sqrt[3]{5 + \sqrt{17}} \times \sqrt[3]{5 - \sqrt{17}} = \sqrt[3]{\left(5 + \sqrt{17}\right)\left(5 - \sqrt{17}\right)}$$

$$= \sqrt[3]{5^2 - 17} = 2.$$

Thus, the equation

$$x^3 = a^3 + b^3 + 3ab(a + b)$$

simplifies to

$$x^3 = 10 + 6x.$$

Hence, x is a root of

$$x^3 - 6x - 10.$$

Problem 16

Answer: **(D)**

Solution 1:

Equating values of y, we get

$$x^2 = 3tx + 4t^2.$$

Rearranging and factoring, we have:

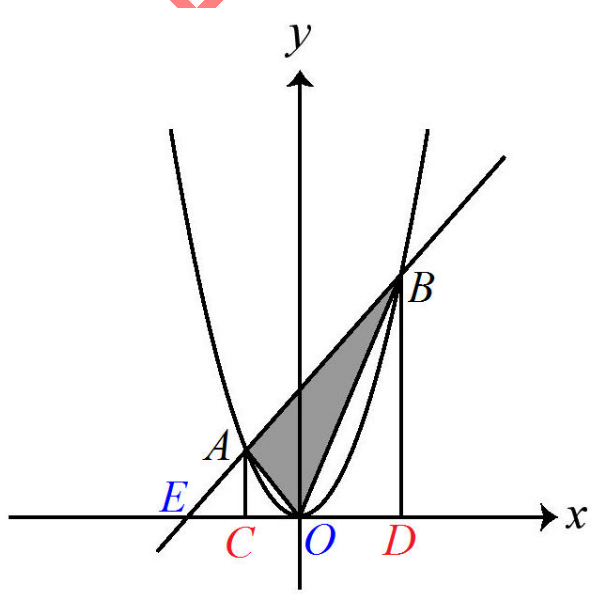
$$(x + t)(x - 4t) = 0,$$

which implies that

$$x = -t, \qquad x = 4t.$$

Thus,

$$A = (-t, t^2), \qquad B = (4t, 16t^2).$$



Using the Shoelace Formula to calculate the area of ΔOAB, we obtain:

$$\text{Aera}(\Delta OAB) = \frac{1}{2}\begin{vmatrix} 0 & 0 \\ -t & t^2 \\ 4t & 16t^2 \\ 0 & 0 \end{vmatrix}$$

$$= \frac{1}{2}\left|\left((-t)\cdot 0 + 4t\cdot t^2 + 0\cdot 16t^2\right) - \left(0\cdot t^2 + (-t)\cdot 16t^2 + 4t\cdot 0\right)\right| = 10t^3.$$

According to the given condition,

$$10t^3 = 640.$$

Hence,

$$t = 4.$$

Solution 2:

Drop perpendiculars from A and B to C and D, respectively, on the x-axis. Then

$$AC = t^2, \qquad BD = 16t^2.$$

Let E be the point of intersection of the line with the x-axis. Setting $3tx + 4t^2 = 0$, we get:

$$x = -\frac{4}{3}t.$$

This means that

$$OE = \frac{4}{3}t.$$

Thus,

$$\text{Aera}(\Delta OAB) = \text{Aera}(\Delta OBE) - \text{Aera}(\Delta OAE) = \frac{OE\cdot BD}{2} - \frac{OE\cdot AC}{2}$$

$$= \frac{\frac{4}{3}t\cdot 16t^2}{2} - \frac{\frac{4}{3}t\cdot t^2}{2} = 10t^3.$$

Because

$$10t^3 = 640,$$

it follows that

$$t = 4.$$

Problem 17

Answer: **(C)**

Note that

$$\frac{100!}{225^a 49^b} = \frac{100!}{3^{2a}5^{2a}7^{2b}}.$$

Since the above expression equals an integer, it follows that each prime factor of the denominator must divide out of the product in the numerator. We count the number of times that each of 7, 5, and 3 is a factor of the numerator.

The number of factors of 7 in 100! is:

$$\left\lfloor\frac{100}{7}\right\rfloor + \left\lfloor\frac{100}{7^2}\right\rfloor = 14 + 2 = 16.$$

The number of factors of 5 in 100! is:

$$\left\lfloor\frac{100}{5}\right\rfloor + \left\lfloor\frac{100}{5^2}\right\rfloor = 20 + 4 = 24.$$

For the numerator to include at least as many factors of 7 as the denominator, we must have

$$16 \geq 2b,$$

or

$$b \leq 8.$$

For the numerator to include at least as many factors of 7 as the denominator, we must have

$$24 \geq 2a,$$

or

$$a \leq 12.$$

Since there are 33 multiples of 3 in the product 100!, the numerator includes at least 33 factors of 3. This means that the number of factors of 2 does not limit the value of a.

Hence, when $a = 12$ and $b = 8$, the given expression is an integer and the maximum value of $a + b$ is

$$12 + 8 = 20.$$

Problem 18

Answer: **(E)**

Let $z = \log_x y$. Note that

$$\log_x y = \frac{1}{\log_y x}.$$

Thus, the given equation can be written as

$$z + \frac{2}{z} = 3,$$

which is equivalent to

$$z^2 - 3z + 2 = 0$$

because $z \neq 0$. Solving for z gives:

$$z = 1, \qquad z = 2.$$

If $z = 1$, then

$$\log_x y = 1,$$

or

$$x = y.$$

Because $2 \leq x \leq 2020$, so there are

$$2020 - 1 = 2019$$

ordered pairs (x, y) such that $x^2 = y$ and x and y satisfy the given conditions.

If $z = 2$, then

$$\log_x y = 2,$$

or

$$x^2 = y.$$

Now $44^2 = 1936$ and $45^2 = 2025$, so there are

$$44 - 1 = 43$$

ordered pairs (x, y) such that $x^2 = y$ and x and y satisfy the given conditions.

Hence, there are

$$2019 + 43 = 2062$$

of the requested ordered pairs. The sum of the digits is:

$$2 + 0 + 6 + 2 = 10.$$

Problem 19

Answer: **(D)**

Let $y = \sin x + \cos x$ and $z = \sin x \cos x$. Then

$$\frac{5}{4} = (1 + \sin x)(1 + \cos x) = 1 + \sin x + \cos x + \sin x \cos x = 1 + y + z,$$

which implies that

$$z = \frac{1}{4} - y.$$

Thus,

$$1 = \sin^2 x + \cos^2 x = y^2 - 2z = y^2 - 2\left(\frac{1}{4} - y\right) = y^2 + 2y - \frac{1}{2}.$$

Solving for y gives:

$$y = -1 \pm \frac{\sqrt{10}}{2}.$$

Because $-2 < y < 2$, it follows that the only possible value for y is:

$$y = -1 + \frac{\sqrt{10}}{2}.$$

Then

$$(1-\sin x)(1-\cos x) = 1 - (\sin x + \cos x) + \sin x \cos x$$

$$= 1 - y + z = \frac{5}{4} - 2y = \frac{5}{4} - 2\left(-1 + \frac{\sqrt{10}}{2}\right)$$

$$= \frac{13}{4} - \sqrt{10}.$$

Hence,

$$a + b + c = 13 + 4 + 10 = 27.$$

Problem 20

Answer: **(D)**

We are given that

$$(x-a)(x-b) = (x+c)(x-6) + 5$$

for all real numbers x.

Substituting $x = 6$ gives

$$(6-a)(6-b) = 5.$$

Since a and b are integers, it follows that $6 - a$ is a divisor of 5. Thus, the possible values of $6 - a$ are

$$\pm 1, \qquad \pm 5.$$

These yield values for a of

$$1, \qquad 5, \qquad 7, \qquad 11.$$

We have to check if each of these values for b gives integer values for b and c.

If $a = 1$, the equation $(6-a)(6-b) = 5$ yields that

$$b = 5.$$

Substituting $a = 1$ and $b = 5$ into the original equation gives

$$(x-1)(x-5)=(x+c)(x-6)+5.$$

Plugging in $x=1$ into the above equation gives:

$$(1+c)(1-6)+5=0,$$

which implies that

$$c=0.$$

Thus

$$(a,b,c)=(1,5,0).$$

If $a=5$, then

$$(a,b,c)=(5,1,0).$$

This is because a and b are interchangeable in the original equation.

Also, if $a=7$, then $b=11$ and we can find that $c=-2$ and

$$(a,b,c)=(7,11,-2).$$

Similarly, if $a=11$, then

$$(a,b,c)=(11,7,-2).$$

Therefore, there are 4integer tuples to satisfy the original equation.

Problem 21

Answer: **(E)**

According to the given conditions,

$$t_1=a, \qquad t_3=b,$$

and for $n\geq 2$

$$t_n=t_{n-1}+t_{n+1}-1.$$

So

$$t_2=t_1+t_3-1=a+b-1.$$

Rearranging gives:

$$t_{n+1}=t_n-t_{n-1}+1.$$

Note that

$$t_4=t_3-t_2+1=b-(a+b-1)+1=2-a,$$

$$t_5 = t_4 - t_3 + 1 = (2-a) - b + 1 = 3 - a - b,$$
$$t_6 = t_5 - t_4 + 1 = (3-a-b) - (2-a) + 1 = 2 - b,$$
$$t_7 = t_6 - t_5 + 1 = (2-b) - (3-a-b) + 1 = a,$$
$$t_8 = t_7 - t_6 + 1 = a - (2-b) + 1 = a + b - 1.$$

Thus,

$$t_7 = t_1 \text{ and } t_8 = t_2.$$

Since each term in the sequence depends only on the previous two terms, it follows that the sequence repeats each 6 terms.

Because

$$2019 = 6 \times 336 + 3,$$

we have:

$$\sum_{k=1}^{2019} t_k = 336 \cdot \left(\sum_{k=1}^{6} t_k\right) + (t_1 + t_2 + t_3)$$
$$= 336 \cdot \big(a + (a+b-1) + b + (2-a) + (3-a-b) + (2-b)\big)$$
$$+ (a + (a+b-1) + b) = 336 \cdot (6) + (2a + 2b - 1)$$
$$= 2016 + (2a + 2b - 1) = 2015 + 2a + 2b.$$

Problem 22

Answer: **(B)**

First observe that if $z \in A$ and $w \in B$, then

$$(zw)^{144} = (z^{18})^8 (w^{48})^3 = 1.$$

This shows that the set C is contained in the set of 144^{th} roots of unity. Next we show that any 144^{th} root of unity is in C, thereby showing that C has 144 elements. Let x be a 144^{th} root of unity. Then there is an integer k with

$$x = \cos\left(\frac{2\pi}{144}k\right) + i \sin\left(\frac{2\pi}{144}k\right) = \text{cis}\left(\frac{2\pi}{144}k\right) = \left[\text{cis}\left(\frac{2\pi}{144}\right)\right]^k,$$

where the last equality follows by an application of DeMoivre's formula. We next express the greatest common divisor of 18 and 48 as $6 = 3 \cdot 18 - 48$ and use this in the following:

$$\operatorname{cis}\left(\frac{2\pi}{144}\right) = \operatorname{cis}\left(\frac{2\pi}{864} \cdot 6\right) = \operatorname{cis}\left(\frac{2\pi}{864}(3 \cdot 18 - 48)\right) = \operatorname{cis}\left(\frac{2\pi}{48}3\right) \operatorname{cis}\left(\frac{2\pi}{18}(-1)\right).$$

By another application of DeMoivre's formula, we now have

$$x = \left[\operatorname{cis}\left(\frac{2\pi}{48}3\right) \operatorname{cis}\left(\frac{2\pi}{18}(-1)\right)\right]^k = \operatorname{cis}\left(\frac{2\pi}{48}3k\right) \operatorname{cis}\left(\frac{2\pi}{18}(-k)\right),$$

which shows that x is a product of elements from A and B. Hence the set of 144^{th} roots of unity is a subset of C. We may conclude that C is the set of 144^{th} roots of unity, so C has 144 elements.

This means that

$$N = 144.$$

Hence, the product of the digits of N is:

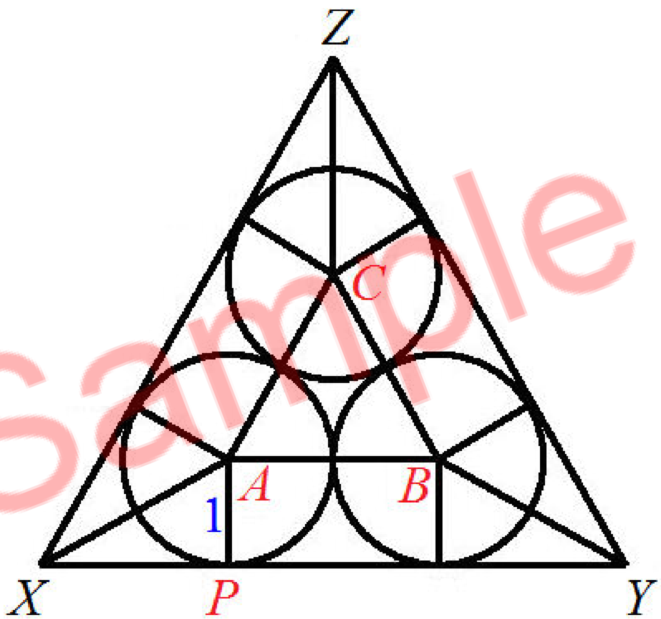
$$1 + 4 + 4 = 9.$$

Problem 23

Answer: **(C)**

To find the volume of the prism, we have to calculate the area of its base and the height of the prism.

First, we calculate the area of its base. Let $A, B,$ and C be the centers of the three mutually tangent spheres, and let $X, Y,$ and Z be the vertices of the triangular cross-section containing $A, B,$ and C, as shown below.



Now let P be the foot of the perpendicular from A to XY. Then ΔAPX is a 30-60-90 triangle with leg $AP = 1$. Thus,

$$XP = \sqrt{3},$$

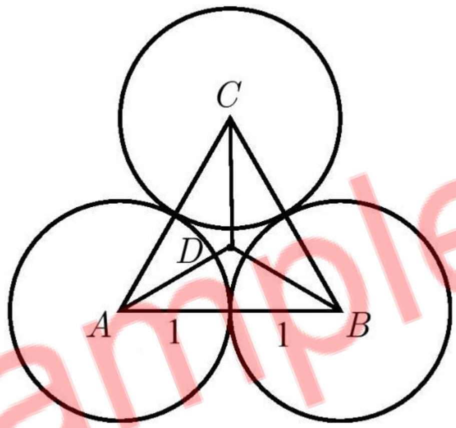
and

$$XY = 2 + 2\sqrt{3}.$$

This means that ΔXYZ is an equilateral triangle with side length $2 + 2\sqrt{3}$. Its area is

$$\text{Aera}(\Delta\text{XYZ}) = \frac{\sqrt{3}}{4}\left(2 + 2\sqrt{3}\right)^2 = 2\left(3 + 2\sqrt{3}\right).$$

Next, we calculate the height of the prism. Let D be the
intersection of the medians of ΔABC, and E be the center of the fourth sphere. Then E is directly above D.



Because $AE = 2$ and $AD = \frac{2}{\sqrt{3}}$, it follows that

$$DE = \sqrt{AE^2 - AD^2} = \sqrt{2^2 - \left(\frac{2}{\sqrt{3}}\right)^2} = \frac{2}{3}\sqrt{6}.$$

This means that the height of the prism is

$$1 + 1 + \frac{2}{3}\sqrt{6} = 2 + \frac{2}{3}\sqrt{6}.$$

Hence, the volume of the prism is

$$2(3 + 2\sqrt{3}) \cdot \left(2 + \frac{2}{3}\sqrt{6}\right) = \frac{4}{3}(3 + 2\sqrt{3})(3 + \sqrt{6}) = 12 + 8\sqrt{2} + 8\sqrt{3} + 4\sqrt{6}.$$

Problem 24

Answer: **(E)**

Solution 1:

Given the values of four rolls, there is exactly one order that satisfies the requirement. So it suffices to count all the sets of values that could be produced by four rolls, allowing duplicate values. This is equivalent to counting the number of ways to put four balls into six boxes labeled 1 through 6. By thinking of 4 balls and 5 dividers to separate the six boxes, this can be seen to be

$$\binom{4+5}{4} = 126.$$

Note that the total number of possible outcome of the four rolls is 6^4. The requested probability is thus

$$\frac{126}{6^4} = \frac{7}{72},$$

so

$$b - a = 72 - 7 = 65.$$

Solution 2:

Let $t_1, t_2, t_3,$ and t_4 be the sequence of values rolled. Consider the difference between the last and the first.

If $t_4 - t_1 = 0$, then there is 1 possibility for t_2 and t_3, and 6 possibilities for t_1 and t_4.

If $t_4 - t_1 = 1$, then there is 3 possibility for t_2 and t_3, and 5 possibilities for t_1 and t_4.

In general, if $t_4 - t_1 = k$, then there is $6 - k$ possibility for t_1 and t_4, while the number of possibilities for t_2 and t_3 is the same as the number of sets of 2 elements, with repetition allowed, that can be chosen from a set of $k + 1$ elements. This is equal to the number of ways to put 2 balls in $k + 1$ boxes, or

$$\binom{k+2}{2}.$$

Thus, there are

$$6 \cdot \binom{2}{2} + 5 \cdot \binom{3}{2} + 4 \cdot \binom{4}{2} + 3 \cdot \binom{5}{2} + 2 \cdot \binom{6}{2} + 1 \cdot \binom{7}{2}$$
$$= 6 \cdot 1 + 5 \cdot 3 + 4 \cdot 6 + 3 \cdot 10 + 2 \cdot 15 + 1 \cdot 21 = 126.$$

sequences of the type requested, so the probability is

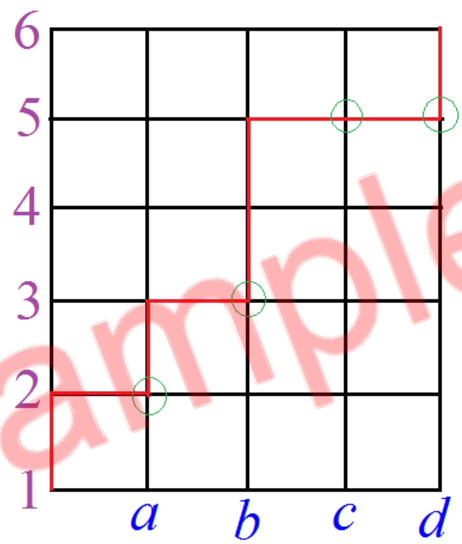
$$\frac{126}{6^4} = \frac{7}{72},$$

and

$$b - a = 72 - 7 = 65.$$

Solution 3:

The problem can be completely recast as a rectangular grid-walking problem. Let $a, b, c,$ and d denote the sequence of values rolled. In the diagram below, the lowest y-coordinate at each of $a, b, c,$ and d corresponds to the value rolled.



The red path corresponds to the sequence 2, 3, 5, 5. This establishes a one-to-one correspondence between valid sequences of values rolled and grid walking paths. Thus, the number of the valid sequences equals the number of the paths from the lower left corner to the upper right corner of the grid, which is:

$$\binom{4+5}{4} = 126.$$

Since the total number of possible outcome of the four rolls is 6^4, the desired probability is

$$\frac{126}{6^4} = \frac{7}{72}.$$

Hence

$$b - a = 72 - 7 = 65.$$

Problem 25

Answer: **(E)**

Note that

$$X_n = \underbrace{xx\cdots x}_{n} = x \cdot 10^{n-1} + x \cdot 10^{n-2} + \cdots + x \cdot 10 + x$$

$$= x \cdot \frac{10^n - 1}{10 - 1} = x \cdot \frac{10^n - 1}{9}.$$

Similarly,

$$Y_n = y \cdot \frac{10^n - 1}{9}, \qquad Z_n = z \cdot \frac{10^{2n} - 1}{9}.$$

The equation $Z_n - Y_n = X_n^2$ is equivalent to

$$z \cdot \frac{10^{2n} - 1}{9} - y \cdot \frac{10^n - 1}{9} = x^2 \left(\frac{10^n - 1}{9}\right)^2.$$

Dividing by $10^n - 1$ and clearing fractions yields

$$9y - 9z - x^2 = (9z - x^2) \cdot 10^n.$$

We consider three cases: $3 \le n \le 2019$, $n = 1$, and $n = 2$.

Case 1: $3 \le n \le 2019$

If $9z - x^2 \neq 0$, then

$$|9z - x^2| \cdot 10^n \ge 1 \cdot 10^3 = 1000.$$

On the other hand,

$$9y - 9z - x^2 \ge 9(0) - 9(9) - 9^2 = -162$$

and

$$9y - 9z - x^2 \le 9(9) - 9(0) - 0^2 = 81.$$

It follows that

$$|9z - x^2| \cdot 10^n = |9y - 9z - x^2| \le 162.$$

Thus, it is impossible that $9z - x^2 \neq 0$.

For $3 \le n \le 2019$, we must have

$$9y - 9z - x^2 = 9z - x^2 = 0,$$

which implies that x is a multiple of 3.

Since x is a positive digit, we have:

$$x = 3, 6, \text{or } 9.$$

If $x = 3$, then

$$9z = x^2 = 9$$

which implies that

$$z = 1$$

and

$$y = z + \frac{x^2}{9} = 1 + 1 = 2.$$

If $x = 6$, then

$$9z = x^2 = 36$$

which implies that

$$z = 4$$

and

$$y = z + \frac{x^2}{9} = 4 + 4 = 8.$$

If $x = 9$, then

$$9z = x^2 = 81$$

which implies that

$$z = 9$$

and

$$y = z + \frac{x^2}{9} = 9 + 9 = 18,$$

which cannot be a digit.

So each n with $3 \leq n \leq 2019$, we obtain two solutions:

$$(n, x, y, z) = (n, 1, 2, 3), (n, 4, 8, 9).$$

Since there are

$$2019 - 3 + 1 = 2017$$

possible values of n, we have

$$2 \times 2017 = 4034$$

quadruples.

Case 1: $n = 1$

We have:

$$9y - 9z - x^2 = 10(9z - x^2),$$

or

$$z = \frac{x^2 + y}{11}.$$

For $x = 1$, there is no solution for (y, z). For each x with $2 \le x \le 9$, there is a solution for (y, z). So we obtain an additional 8 quadruples.

Case 3: $n = 2$

We have:

$$9y - 9z - x^2 = 100(9z - x^2),$$

or

$$z = \frac{11x^2 + y}{101}.$$

Using the trial and error approach to check $x = 1, 2, \cdots, n$, we find 3 solutions:

$$(x, y, z) = (3, 2, 1), (6, 8, 4), (8, 3, 7).$$

Thus, we obtain an additional 3 quadruples.

Hence, in total, there are

$$4034 + 8 + 3 = 4045$$

quadruples.