AIME Prep Course

(Tutorial Handout Sample)

***Topic*:**

Algebraic Equations

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Contents

1. Introduction

Many problems in the AIME competition require solving systems of equations. When dealing with a system of equations there are two standard methods to algebraically solve the system. One is substitution and the other is elimination. However, it is difficult to directly apply substitution or elimination to solve many systems of equations problems on the AIME. We first need to find suitable algebraic transformations to convert complicated equations to simpler equations. Using some typical examples, we will demonstrate how to quickly develop good algebraic transformations to simplify complex systems of equations.

In most cases, equations and systems of equations are meant to be solved. After all, if we could pinpoint down the exact values of the variables that can be plugged in to make the equality true, then we have in effect completely understood the equation at hand. However, sometimes these values are infeasible to compute by hand and sometimes they are not even important to what you're trying to do! In these situations, algebraic finesse is crucial to success. We will explore problems that require creativity and show that sometimes finding exact values of variables is not the best way to go.

Throughout this handout, you will see examples of problems where greatly taking advantage of symmetry is key. This is because symmetry is nice enough that it often helps with exploiting patterns -- and patterns are often the driving force behind making complicated problems simple.

2. Important Formulas

Many times, solving an algebraic manipulation problem will boil down to recognizing one or several common factorizations and manipulations. Here are just a few; you will find some others in the exercises.

- $(x+y)^2 = x^2 + 2xy + y^2$.
- $(x-y)^2 = x^2 - 2xy + y^2$.
- $(x+y+z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + zx)$.
- $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 = x^3 + y^3 + 3xy(x+y)$.
- $(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3 = x^3 - y^3 - 3xy(x-y)$.
- $(x+y)^n$
$$= \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}x^1 y^{n-1} + \binom{n}{n}x^0 y^n$$
$$= \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}.$$
- $(x_1 + x_2 + \cdots + x_m)^n$
$$= \sum_{k_1+k_2+\cdots+k_m=n}\binom{n}{k_1, k_2, \cdots, k_m} x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m},$$
where
$$\binom{n}{k_1, k_2, \cdots, k_m} = \frac{n!}{k_1!\, k_2! \cdots k_m!}$$
- $x^2 - y^2 = (x-y)(x+y)$.
- $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + y^{n-1})$ for all n.
- $a^{2m+1} + b^{2m+1} = (a+b)(a^{2m} - a^{2m-1}b + a^{2m-2}b^2 - \cdots + b^{2m})$ (the terms of the second factor alternate in sign).
- $a + b + c + ab + bc + ca + abc = (a+1)(b+1)(c+1)$.
- $a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ac)$.
- $(a+b+c)^3 = a^3 + b^3 + c^3 + 3a^2b + 3a^2c + 3ab^2 + 3b^2c + 3ac^2 + 3bc^2 + 6abc$

$$= a^3 + b^3 + c^3 + 3(a+b+c)(ab+bc+ca) - 3abc.$$

- $a^4 + 4b^4 = (a^2 + 2b^2 - 2ab)(a^2 + 2b^2 + 2ab)$ (Sophie-Germain Identity)
- $x^4 + x^2 + 1 = (x^2 + x + 1)(x^2 - x + 1)$

The most valuable technique, however, is simple: just try stuff! The answer will rarely pop out immediately; you will often need to try different things and stumble a bit along the way. Even still, don't try things just at random -- instead, focus on improving your position on the battlefield. This is where practice comes in, for it allows one to develop the kind of thinking that leads to these conclusions more quickly.

3. Problem Solving

The AIME commonly contains problems involving systems of equations, but the systems can often be much more complicated, involving numerous equations that are sometimes nonlinear. We usually use the division and multiplication properties of equality to solve equations. In addition, many of the problems have the property that the solution is the same even if the names of the variables are permuted. Symmetry analysis can be used to simplify the analysis of these problems and provides the most important method for solving the "symmetric equations."

It takes practice and experience with problems involving systems of equations to successfully solve them; few specific guidelines are available, and a bit of cleverness is often required. Let us explore a couple of examples.

Example 1: 1999 Junior Balkan Mathematical Olympiad (JBMO) #1

Let a, b, c, x, y be five real numbers such that

$$a^3 + ax + y = 0, \ \ b^3 + bx + y = 0, \ \text{ and } \ c^3 + cx + y = 0.$$

If a, b, c are all distinct numbers prove that their sum is zero.

Solution

The equations can be rewritten as:

$$a^3 + ax = -y,$$

$$b^3 + bx = -y,$$

$$c^3 + cx = -y.$$

Subtracting the second equation from the first gives:

$$a^3 + ax - (b^3 + bx) = 0.$$

This factors as:

$$(a-b)(a^2 + ab + b^2 + x) = 0.$$

Since $a \neq b$, it follows that

$$a^2 + ab + b^2 = -x.$$

Similarly,

$$a^2 + ac + bc^2 = -x.$$

Thus,

$$a^2 + ab + b^2 = a^2 + ac + c^2,$$

or

$$ab - ac + b^2 - c^2 = 0.$$

This factors as:

$$(a + b + c)(b - c) = 0.$$

Note that $b \neq c$. Hence,

$$a + b + c = 0.$$

Example 2.   2000 AIME I #7

Suppose that $x,\ y,$ and z are three positive numbers that satisfy the equations

$$xyz = 1,\ \ x + \frac{1}{z} = 5,\ \ \text{and}\ \ y + \frac{1}{x} = 29.$$

Then

$$z + \frac{1}{y} = \frac{m}{n},$$

where m and n are relatively prime positive integers. Find $m + n$.

Answer: 005

Solution 1

Notice that

$$5 \cdot 29 \cdot \frac{m}{n} = \left(x + \frac{1}{z}\right)\left(y + \frac{1}{x}\right)\left(z + \frac{1}{y}\right)$$

$$= xyz + x + \frac{1}{z} + y + \frac{1}{x} + z + \frac{1}{y} + \frac{1}{xyz}$$

$$= 1 + 5 + 29 + \frac{m}{n} + 1.$$

Thus,

$$144 \cdot \frac{m}{n} = 36,$$

so that

$$\frac{m}{n} = \frac{1}{4}.$$

Hence,

$$m + n = 1 + 4 = 5.$$

Solution 2:

Multiplying by x both sides of the third equation gives:

$$xy + 1 = 29x,$$

so

$$xy = 29x - 1.$$

Substituting this into the first equation yields:

$$(29x - 1)z = 1,$$

so

$$\frac{1}{z} = 29x - 1.$$

Substituting this into the second equation gets:

$$x + (29x - 1) = 5,$$

which implies that

$$x = \frac{1}{5}.$$

So,

$$y = 29 - \frac{1}{\frac{1}{5}} = 24$$

and

$$z = \frac{1}{xy} = \frac{1}{\frac{1}{5} \cdot 24} = \frac{5}{24}.$$

Hence,

$$z+\frac{1}{y}=\frac{5}{24}+\frac{1}{24}=\frac{1}{4}$$

and

$$m+n=1+4=5.$$

Example 3. **2012 AIME II #8**

The complex numbers z and w satisfy the system

$$z+\frac{20i}{w}=5+i,$$

$$w+\frac{12i}{z}=-4+10i.$$

Find the smallest possible value of $|zw|^2$.

Answer: **040**

Solution:

Multiplying corresponding sides of the two equations gives

$$zw+32i-\frac{240}{zw}=-30+46i$$

and multiplying by zw then yields a quadratic in zw:

$$(zw)^2+(30-14i)zw-240=0.$$

Using the quadratic formula, we find the two possible values of zw to be

$$zw=7i-15\pm\sqrt{(15-7i)^2+240}=7i-15\pm\sqrt{416-210i}.$$

Now let

$$416-210i=(a+bi)^2.$$

Equating the real and imaginary parts results in

$$a^2-b^2=416,\ \ ab=-105.$$

The real solutions to this system are

$$(a,b)=(21,-5)\quad \text{or}\quad (-21,5).$$

Thus,

$$zw=7i-15\pm(21-5i),$$

so

$$zw=6+2i\ \ \text{or}\ \ -36+12i.$$

Thus, the smallest possible value of $|zw|^2$ is

$$6^2 + 2^2 = 40 = 40.$$

This value is attained when

$$w = 2 + 4i \quad \text{and} \quad z = 1 - i.$$

Next, we study an example that consists of a larger system, but consisting of linear equations.

Example 4. 1989 AIME #8

Assume that $x_1, x_2, \cdots, x_7$ are real numbers such that

$$x_1 + 4x_2 + 9x_3 + 16x_4 + 25x_5 + 36x_6 + 49x_7 = 1,$$

$$4x_1 + 9x_2 + 16x_3 + 25x_4 + 36x_5 + 49x_6 + 64x_7 = 12,$$

$$9x_1 + 16x_2 + 25x_3 + 36x_4 + 49x_5 + 64x_6 + 81x_7 = 123.$$

Find the value of

$$16x_1 + 25x_2 + 36x_3 + 49x_4 + 64x_5 + 81x_6 + 100x_7 .$$

Answer: 334

Solution:

Observe that subtracting 3 times the second equation from the sum of the first equation and 3 times the thirst equation gives an equation with the desired expression on the left. Thus,

$$16x_1 + 25x_2 + 36x_3 + 49x_4 + 64x_5 + 81x_6 + 100x_7$$

$$= 1 \cdot 1 - 3 \cdot 12 + 3 \cdot 123 = 334.$$

More formally, we can deduce the relation mentioned above by finding constraints, $a, b,$ and c such that

$$an^2 + b(n+1)^2 + c(n+2)^2 = (n+3)^2.$$

hold for all n. Considering this equation as a polynomial identity in n, we expand and simplify both sides, then equate coefficients of like powers of n. We obtain the equations:

$$a + b + c = 1,$$

$$2b + 4c = 6,$$

$$b + 4c = 9.$$

The solution of this system is

$$a = 1,\ \ b = -3,\ \ \text{and}\quad c = 3.$$

Example 5: **1990 AIME #15**

Find $ax^5 + by^5$ if the real numbers $a, b, x,$ and y satisfy the equations

$$ax + by = 3,$$

$$ax^2 + by^2 = 7,$$

$$ax^3 + by^3 = 16,$$

$$ax^4 + by^4 = 42.$$

Answer: **20**

Solution 1:

We have the identity formula:

$$(ax^{n+1} + by^{n+1})(x + y) - (ax^n + by^n)xy = ax^{n+2} + by^{n+2}$$

for integer $n \geq 1$.

For $n = 1$ and $n = 2$, the identity yields:

$$7(x + y) - 3xy = 16,$$

$$16(x + y) - 7xy = 42.$$

Solving these two equations simultaneously, gives:

$$x + y = -14\ \ \text{and}\ \ xy = -38.$$

Thus, using the identity formula with $n = 3$ yields:

$$ax^5 + by^5 = (ax^4 + by^4)(x + y) - (ax^3 + by^3)xy$$

$$= (42)(-14) - (16)(-38) = -588 + 608 = 20.$$

Note:

From $x + y = -14$ and $xy = -38$, we can solve for x and y to obtain

$$x = -7 \pm \sqrt{87} \text{ and } y = -7 \mp \sqrt{87},$$

from which

$$a = \frac{49}{76} \pm \frac{457}{6612}\sqrt{87} \text{ and } b = \frac{49}{76} \mp \frac{457}{6612}\sqrt{87}.$$

Solution 2:

Subtracting x times the first equation from the second yields

$$by(y - x) = 7 - 3x. \qquad (1)$$

Subtracting x times the second equation from the third yields

$$by^2(y - x) = 16 - 7x. \qquad (2)$$

Subtracting x times the third equation from the fourth yields

$$by^3(y - x) = 42 - 16x. \qquad (3)$$

Dividing Equation (2) by Equation (1) gives:

$$y = \frac{16 - 7x}{7 - 3x}.$$

Dividing Equation (3) by Equation (2) gives:

$$y = \frac{42 - 16x}{16 - 7x}.$$

Thus,

$$\frac{16-7x}{7-3x}=\frac{42-16x}{16-7x},$$

which reduces to

$$x^2+14x-38=0.$$

Completing the square gives:

$$(x+7)^2=87,$$

which implies that

$$x=-7\pm\sqrt{87}.$$

Thus,

$$y=-7\mp\sqrt{87}.$$

Now let

$$ax^5+by^5=k.$$

Subtracting x times the original fourth equation from the equation above yields

$$by^4(y-x)=k-42x. \qquad (4)$$

Dividing Equation (4) by Equation (3) gives:

$$y=\frac{k-42x}{42-16x},$$

which implies that

$$k=42x+(42-16x)y.$$

Hence,

$$k=42\left(-7+\sqrt{87}\right)+\left(42-16\left(-7+\sqrt{87}\right)\right)\left(-7-\sqrt{87}\right)=20.$$

Generalization:

If the real numbers $a, b, c, x, y,$ and z satisfy the equations

$$ax + by + cz = 1,$$

$$ax^2 + by^2 + cz^2 = 2,$$

$$ax^3 + by^3 + cz^3 = 3,$$

$$ax^4 + by^4 + cz^4 = 5,$$

$$ax^5 + by^5 + cz^5 = 8,$$

what is the value of $ax^6 + by^6 + cz^6$.

Let

$$f_n = ax^n + by^n + cy^n,$$

for $n \geq 1$. Then

$$f_1 = 1,\ \ f_2 = 2,\ \ f_3 = 3,\ \ f_4 = 5,\ \ f_5 = 8.$$

Example 6: **2015 AIME II #14**

Let x and y be real numbers satisfying

$$x^4y^5 + y^4x^5 = 810$$

and

$$x^3y^6 + y^3x^6 = 945.$$

Evaluate $2x^3 + (xy)^3 + 2y^3$.

Answer: **089**

Solution 1

Factoring gives:

$$x^4y^4(x+y) = 810$$

and

$$x^3y^3(x+y)(x^2-xy+y^2) = 945.$$

Dividing the second equation by the first gives

$$\frac{x^2-xy+y^2}{xy} = \frac{945}{810} = \frac{7}{6}.$$

Adding 3 to both sides and simplifying yields

$$\frac{(x+y)^2}{xy} = \frac{25}{6}.$$

Thus,

$$x+y = \frac{5}{\sqrt{6}}\sqrt{xy}$$

and

$$xy = \frac{6}{25}(x+y)^2.$$

Substituting $x+y = \frac{5}{\sqrt{6}}\sqrt{xy}$ into the first equation yields

$$\frac{5\sqrt{6}}{6}(xy)^{\frac{9}{2}} = 810.$$

Solving for xy gets

$$xy = 3\sqrt[3]{2},$$

so

$$x^3y^3 = 54.$$

Substituting this into the second equation

$$54(x^3+y^3) = 945,$$

which implies that

$$x^3 + y^3 = \frac{35}{2}.$$

Hence,

$$2x^3 + (xy)^3 + 2y^3 = 2\left(\frac{35}{2}\right) + 54 = 89.$$

Solution 2

Factoring gives:

$$x^4y^4(x + y) = 810$$

and

$$x^3y^3(x + y)(x^2 - xy + y^2) = 945.$$

Dividing the second equation by the first gives

$$\frac{x^2 - xy + y^2}{xy} = \frac{945}{810} = \frac{7}{6},$$

or

$$\frac{x}{y} - 1 + \frac{y}{x} = \frac{7}{6}.$$

Let $\frac{y}{x} = t$. Then the equation above can be converted to the following quadratic

$$6t^2 - 13t + 6 = 0,$$

which can be factored into

$$(2t - 3)(3t - 2) = 0.$$

There are two solutions for t:

$$t = \frac{3}{2} \quad \text{or} \quad \frac{2}{3}.$$

Because the original two equations are symmetric in x and y, we only need to consider the case when

$$\frac{y}{x} = \frac{3}{2}.$$

That is,

$$y = \frac{3}{2}x\,.$$

Now plugging this into the original first equation gets

$$x^4\left(\frac{3}{2}x\right)^5 + \left(\frac{3}{2}x\right)^4 x^5 = 810,$$

which reduces to

$$x^9 = 2^6.$$

So

$$x^3 = 4 \text{ and } y^3 = \frac{27}{2}.$$

Hence,

$$2x^3 + (xy)^3 + 2y^3 = 2 \cdot 4 + 4 \cdot \frac{27}{2} + 2 \cdot \frac{27}{2} = 89.$$

Solution 3

Let

$$a = xy \quad \text{and} \quad b = x + y.$$

Then the first equation becomes

$$a^4 b = 810. \tag{1}$$

Note that

$$x^3y^6 + y^3x^6 = x^3y^3(x^3 + y^3) = x^3y^3((x + y)^3 - 3xy(x + y)).$$

Thus, the second equation can be rewritten as:

$$a^3(b^3 - 3ab) = 945. \tag{2}$$

Dividing Equation (2) by Equation (1) gets:

$$\frac{b^2 - 3a}{a} = \frac{7}{6}$$

or

$$\frac{b^2}{a} = \frac{25}{6}.$$

Thus,

$$a = \frac{6}{25}b^2.$$

Substituting this in Equation (2) yields:

$$b^3 = \frac{5^3}{2},$$

which implies that

$$b = \frac{5}{\sqrt[3]{2}}.$$

So

$$a = \frac{6}{25}\left(\frac{5}{\sqrt[3]{2}}\right)^2 = \frac{6}{\sqrt[3]{4}}.$$

Hence,

$$2x^3 + (xy)^3 + 2y^3 = (xy)^3 + 2(x^3 + y^3)$$

$$= (xy)^3 + 2((x + y)^3 - 3xy(x + y))$$

$$= a^3 + 2b^3 - 6ab = \left(\frac{6}{\sqrt[3]{4}}\right)^3 + 5^3 - 6 \cdot \frac{6}{\sqrt[3]{4}} \cdot \frac{5}{\sqrt[3]{2}}$$

$$= 54 + 125 - 90 = 89.$$

Solution 4

Adding three times the first equation to the second and then factoring gives:

$$(xy)^3(x^3 + 3x^2y + 3xy^2 + y^3) = (xy)^3(x + y)^3 = 3375.$$

Taking the cube root yields

$$xy(x + y) = 15.$$

Note that the first equation can be rewritten as:

$$(xy)^3\big(xy(x + y)\big) = 810.$$

Thus,

$$(xy)^3(15) = 810,$$

which implies that

$$(xy)^3 = 54.$$

Plugging this into the second equation gives:

$$54(x^3 + y^3) = 945,$$

which implies that

$$x^3 + y^3 = \frac{35}{2}.$$

Hence,

$$2x^3 + (xy)^3 + 2y^3 = (xy)^3 + 2(x^3 + y^3) = 54 + 2\left(\frac{35}{2}\right) = 89.$$

Example 7: 2021 AIME II #7

Let $a, b, c,$ and d be real numbers that satisfy the system of equations

$$a + b = -3,$$

$$ab + bc + ca = -4,$$

$$abc + bcd + cda + dab = 14,$$

$$abcd = 30.$$

There exist relatively prime positive integers m and n such that

$$a^2 + b^2 + c^2 + d^2 = \frac{m}{n}.$$

Answer: 145

Solution:

Factoring d out of the third equation gets

$$abc + d(bc + ac + ab) = 14.$$

Using the original second equation, we know that the expression in the parentheses is -4. So

$$abc - 4d = 14.$$

From the original fourth equation, we get

$$abc = \frac{30}{d}.$$

Substituting this into the equation above gives:

$$\frac{30}{d} - 4d = 14.$$

This simplifies to

$$2d^2 + 7d - 15 = 0.$$

Factoring the quadratic gives

$$(d + 5)(2d - 3) = 0,$$

so the solutions occur at $d = -5$ and $d = \frac{3}{2}$.

If $d = -5$, then

$$abc = \frac{30}{d} = -6.$$

The system becomes:

$$a+b=-3,$$
$$ab+bc+ca=-4,$$
$$abc=-6.$$

We use the top equation to change the middle one to

$$ab+c(a+b)=ab-3c=-4.$$

And then we use $ab=-\frac{6}{c}$ from the bottom equation to get

$$-\frac{6}{c}-3c=-4,$$

which simplifies to

$$3c^2-4c+6=0.$$

Its discriminant is

$$4^2-4\cdot 3\cdot 6=16-72<0.$$

Thus, there are no real solutions in this case.

If $d=\frac{3}{2}$, then

$$abc=\frac{30}{d}=20.$$

The original system becomes

$$a+b=-3,$$
$$ab+bc+ca=-4,$$
$$abc=20.$$

Factoring d out of the second equation gets

$$ab+c(a+b)=-4.$$

Plugging the first equation into the equation above yields:

$$ab-3c=-4.$$

Then substituting $ab=\frac{20}{c}$ from the third equation gives:

$$\frac{20}{c} - 3c = -4,$$

which simplifies to

$$3c^2 - 4c - 20 = 0.$$

This factors as

$$(3c - 10)(c + 2) = 0.$$

So

$$c = \frac{10}{3} \quad \text{or} \quad c = -2.$$

If we use $c = \frac{10}{3}$, the we have:

$$a + b = -3,$$

$$ab = 6.$$

Plugging in $a = \frac{6}{b}$ into the top gives

$$\frac{6}{b} + b = -3.$$

This is the quadratic

$$b^2 + 3b + 6 = 0.$$

This has discriminant

$$3^2 - 4 \cdot 6 = -15 < 0,$$

so it doesn't have real solutions.

Thus, it must be the case that $c = -2$. This gives us the system:

$$a + b = -3,$$

$$ab = -10.$$

Plugging in $a = -\frac{10}{b}$ into the top gives

$$-\frac{10}{b} + b = -3.$$

This simplifies to

$$b^2 + 3b - 10 = 0,$$

which factors as

$$(b+5)(b-2)=0,$$

so

$$b=-5 \quad \text{or} \quad b=2.$$

The system is however symmetric in a and b, so we can switch a and b such that $b=-5$ gives $a=2$ and $a=-5$ gives $b=2$.

Thus, we have two solutions:

$$(a,b,c,d)=\left(-5,2,-2,\frac{3}{2}\right),\ \ (a,b,c,d)=\left(2,-5,-2,\frac{3}{2}\right).$$

Hence,

$$a^2+b^2+c^2+d^2=(-2)^2+5^2+(-2)^2+\left(\frac{3}{2}\right)^2=\frac{141}{4}$$

and

$$m+n=141+4=145.$$

Example 8. 2006 AIME II #15

Given that $x,\ y,$ and z are real numbers that satisfy:

$$x=\sqrt{y^2-\frac{1}{16}}+\sqrt{z^2-\frac{1}{16}},$$

$$y=\sqrt{z^2-\frac{1}{25}}+\sqrt{x^2-\frac{1}{25}},$$

$$z=\sqrt{x^2-\frac{1}{36}}+\sqrt{y^2-\frac{1}{36}},$$

and that

$$x+y+z=\frac{m}{\sqrt{n}},$$

where m and n are positive integers and n is not divisible by the square of any prime, find $m+n$.

Answer: **009**

Solution:

Solution 1 (Geometric Interpretation)

The radicals on the right side of the first equation are reminiscent of the Pythagorean Theorem. Each radical represents the length of a leg of a right triangle whose other leg has length $\frac{1}{4}$, and whose hypotenuse has length y or z.

Adjoin these two triangles along the leg of length $\frac{1}{4}$ to create ΔXYZ with

$$x = YZ, \quad y = ZX, \quad \text{and} \quad z = XY,$$

and with altitude to side YZ of length $\frac{1}{4}$. Because of similar considerations in the other two equations, let the lengths of the altitudes to sides ZX and XY be $\frac{1}{5}$ and $\frac{1}{6}$, respectively.

Note that in the ΔXYZ, the lengths $x, y,$ and z of the sides satisfy the given equations, provided the altitudes of ΔXYZ are inside the triangle, that is, provided ΔXYZ is acute.

In general, a triangle the lengths of whose sides are $a, b,$ and c is acute if and only if

$$a^2 + b^2 > c^2,$$

where $a \le b \le c$.

Denote the area of the triangle by K and the lengths of the altitudes to the sides of lengths $a, b,$ and c by $h_a, h_b,$ and h_c, respectively. Then

By

$$K = \frac{1}{2}ah_a = \frac{1}{2}bh_b = \frac{1}{2}ch_c,$$

so the condition $a^2 + b^2 > c^2$ is equivalent to

$$\left(\frac{1}{h_a}\right)^2 + \left(\frac{1}{h_b}\right)^2 > \left(\frac{1}{h_c}\right)^2,$$

where

$$\frac{1}{h_a} \le \frac{1}{h_b} \le \frac{1}{h_c}.$$

Thus, ΔXYZ is acute because

$$4^2 + 5^2 > 6^2.$$

Let K be the area of ΔXYZ. Then

$$x = 8K, \;\; y = 10K, \;\; \text{and} \;\; z = 12K,$$

so

$$x + y \; + z = 30K.$$

Apply Heron's Formula to obtain

$$K^2 = 15K \cdot 7K \cdot 5K \cdot 3K.$$

Because $K > 0$, it follows that

$$K = \frac{1}{15\sqrt{7}} \, .$$

Then

$$x + y + z = 30K = \frac{2}{\sqrt{7}},$$

so

$$m + n = 2 + 7 = 9.$$

Solution 2 (Algebraic Method)

The first equation can be rewritten as:

$$\sqrt{y^2 - \frac{1}{16}} = x - \sqrt{z^2 - \frac{1}{16}}.$$

Squaring both sides gives:

$$y^2 - \frac{1}{16} = x^2 - 2x\sqrt{z^2 - \frac{1}{16}} + z^2 - \frac{1}{16}.$$

Isolating the radical gets:

$$2x\sqrt{z^2 - \frac{1}{16}} = x^2 - y^2 + z^2 \, .$$

Squaring both sides yields:

$$4x^2\left(z^2 - \frac{1}{16}\right) = x^4 + y^4 + z^4 - 2x^2y^2 - y^2z^2 + 2x^2z^2.$$

This simplifies to:

$$x^4 + y^4 + z^4 - 2x^2y^2 - 2y^2z^2 - 2x^2z^2 = -\frac{1}{16} \cdot 4x^2. \tag{1}$$

Similarly, from the second and third equations, we get

$$x^4 + y^4 + z^4 - 2x^2y^2 - 2y^2z^2 - 2x^2z^2 = -\frac{1}{25} \cdot 4y^2, \tag{2}$$

and

$$x^4 + y^4 + z^4 - 2x^2y^2 - 2y^2z^2 - 2x^2z^2 = -\frac{1}{36} \cdot 4z^2. \tag{3}$$

Thus,

$$x : y : z = 4 : 5 : 6,$$

which implies that

$$x = \frac{4}{5}y \quad \text{and} \quad z = \frac{6}{5}y.$$

Equation (2) can be rewritten as:

$$y^2\left(y^2 - 2x^2 - 2z^2 + \frac{4}{25}\right) + (x^2 - z^2)^2 = 0.$$

Plug the expressions of x and z into this equation to get

$$y^2\left(y^2 - 2\left(\frac{4}{5}y\right)^2 - 2\left(\frac{6}{5}y\right)^2 + \frac{4}{25}\right) + \left(\left(\frac{4}{5}y\right)^2 - \left(\frac{6}{5}y\right)^2\right)^2 = 0.$$

which simplifies to:

$$63y^4 - 4y^2 = 0.$$

But $y \neq 0$. So

$$y^2 = \frac{4}{63}$$

or

$$y = \frac{2}{3\sqrt{7}}.$$

Hence,

$$x+y+z=\frac{4}{5}y+y+\frac{6}{5}y=3y=\frac{2}{\sqrt{7}}$$

and

$$m+n=2+7=9.$$

Example 9. **2022 AIME I #15**

Let $x,\ y,$ and z be positive real numbers satisfying the system of equations:

$$\sqrt{2x-xy}+\sqrt{2y-xy}=1,$$
$$\sqrt{2y-yz}+\sqrt{2z-yz}=\sqrt{2},$$
$$\sqrt{2z-zx}+\sqrt{2x-zx}=\sqrt{3}.$$

Then $[(1-x)(1-y)(1-z)]^2$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Answer: **033**

Solution 1:

The square root of any real number is either real or pure imaginary, while the right side of each of the given equations is a nonzero real number. Thus, in each of the equations both square roots evaluate to a real number and each of the six expressions under the radicals is nonnegative. This in turn implies that

$$0\le x,\ \ y,\ \ z\le 2.$$

Let $a=1-x,\ b=1-y,\ c=1-z$, with $-1\le a,\ b,\ c\le 1$. Then the requested expression is equal to $(abc)^2$; and the system becomes

$$\sqrt{(1-a)(1+b)}+\sqrt{(1+a)(1-b)}=1,$$
$$\sqrt{(1-b)(1+c)}+\sqrt{(1+b)(1-c)}=\sqrt{2},$$
$$\sqrt{(1-a)(1+c)}+\sqrt{(1+a)(1-c)}=\sqrt{3}.$$

Squaring the equations and simplifying gives

$$2\sqrt{(1-a^2)(1-b^2)}=2ab-1,$$
$$\sqrt{(1-b^2)(1-c^2)}=bc,$$
$$2\sqrt{(1-a^2)(1-c^2)}=2ac+1,$$

Note that it is necessary that $2ab-1, bc,$ and $2ac+1$ are nonnegative. Squaring and simplifying again gives

$$4a^2+4b^2-4ab=3,$$
$$b^2+c^2=1,$$
$$4a^2+4c^2+4ac=3.$$

Subtracting the third equation from the first yields

$$b^2-c^2-ab-ac=(b+c)(b-a-c)=0.$$

The case $b=-c$ is incompatible with the condition

$$\sqrt{(1-b^2)(1-c^2)}=bc.$$

Thus, the system of equations simplifies to

$$a+c=b,$$
$$b^2+c^2=1,$$
$$a^2+c^2+ac=\frac{3}{4}.$$

Substituting the first equation into the second gives

$$a^2+2c^2+2ac=1,$$

which, combined with the third equation yields

$$a^2=\frac{1}{2},$$

The condition $b=a+c$ implies that

$$(b-c)^2=b^2+c^2-2bc=a^2=\frac{1}{2},$$

from which

$$bc=\frac{1}{4}.$$

Hence,

$$(abc)^2=(bc)^2\cdot a^2=\frac{1}{32}$$

and the requested sum is

$$1+32=33.$$

The conditions are satisfied by

$$(a,b,c)=\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2\sqrt{2}+1}}{2},\frac{1}{2\sqrt{2\sqrt{2}+1}}\right).$$

Solution 2:

The square root of any real number is either real or pure imaginary, while the right side of each of the given equations is a nonzero real number. Thus, in each of the equations both square roots evaluate to a real number and each of the six expressions under the radicals is nonnegative. This in turn implies that

$$0\le x,\ \ y,\ \ z\le 2.$$

So, there exist $\alpha,\ \beta,$ and γ with $0\le\alpha,\ \beta,\ \gamma\le 90°$ such that

$$x=2\sin^2\alpha\,,\ \ y=2\sin^2\beta\,,\ \ \text{and}\ \ z=2\sin^2\gamma.$$

Then

$$\sqrt{2x-xy}+\sqrt{2y-xy}=2\sqrt{\sin^2\alpha\cos^2\beta}+2\sqrt{\cos^2\alpha\sin^2\beta}=2\sin(\alpha+\beta).$$

After a similar substitution into the second and third equations, the system becomes

$$\sin(\alpha+\beta)=\frac{1}{2},$$

$$\sin(\beta+\gamma)=\frac{\sqrt{2}}{2},$$

$$\sin(\alpha+\gamma)=\frac{\sqrt{3}}{2}.$$

Thus,

$$\alpha+\beta=30°\ \ \text{or}\ \ 150°,\ \ \beta+\gamma=\ 45°\ \ \text{or}\ \ 135°,\ \ \text{and}\ \ \alpha+\gamma=60°\ \ \text{or}\ \ 120°.$$

Note that if $\alpha,\ \beta,$ and γ satisfy these conditions, then so do $90°-\alpha,\ 90°-\beta,$ and $90°-\gamma$.

So, it is sufficient to consider the case when $\alpha+\beta=30°$.

Because $\gamma\le 90°$, it follows that in this case

$$\beta+\gamma=\ 45°\ \ \text{and}\ \ \alpha+\gamma=60°.$$

Thus, the only two systems of equations that have solutions under the constraints $0\le\alpha,\ \beta,\ \gamma\le 90°$ are

$$\alpha+\beta=30°,$$

$$\beta+\gamma=\ 45°,$$

$$\alpha + \gamma = 60°;$$

or

$$\alpha + \beta = 150°,$$
$$\beta + \gamma = 135°,$$
$$\alpha + \gamma = 120°.$$

That is,

$$(\alpha, \beta, \gamma) = (22.5°, 7.5°, 37.5°) \quad \text{or} \quad (67.5°, 82.5°, 52.5°).$$

Note that

$$(1-x)(1-y)(1-z) = (1 - 2\sin^2\alpha)(1 - 2\sin^2\beta)(1 - 2\sin^2\gamma)$$
$$= \cos(2\alpha)\cos(2\beta)\cos(2\gamma).$$

In the first case,

$$(1-x)(1-y)(1-z) = \cos(45°)\cos(15°)\cos(75°)$$
$$= \cos(45°)\cos(15°)\sin(15°)$$
$$= \cos(45°) \cdot \frac{\sin(30°)}{2} = \frac{\sqrt{2}}{8}.$$

In the second case,

$$(1-x)(1-y)(1-z) = \cos(135°)\cos(165°)\cos(105°)$$
$$= -\cos(45°)\cos(15°)\cos(75°)$$
$$= -\frac{\sqrt{2}}{8}.$$

Hence in both cases,

$$[(1-x)(1-y)(1-z)]^2 = \left(\frac{\sqrt{2}}{8}\right)^2 = \frac{1}{32}.$$

The requested sum is

$$1 + 32 = 33.$$

CLASSWORK

Exercise Set

Exercise 1:

If a, b, and c are positive numbers satisfying

$$a+\frac{1}{b}=1,\ \ b+\frac{1}{c}=\frac{10}{3},\ \text{ and }\ \ c+\frac{1}{a}=\frac{11}{4}.$$

Then the sum of all possible values of abc can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Exercise 2: 1986 AIME #4

Determine $3x_4+2x_5$ if x_1, x_2, x_3, x_4, and x_5 satisfy the system of equations below.

$$2x_1+x_2+x_3+x_4+x_5=6,$$
$$x_1+2x_2+x_3+x_4+x_5=12,$$
$$x_1+x_2+2x_3+x_4+x_5=24,$$
$$x_1+x_2+x_3+2x_4+x_5=48,$$
$$x_1+x_2+x_3+x_4+2x_5=96.$$

Exercise 3:

Suppose w, x, y, and z satisfy

$$3w+x+y+z=20,$$
$$w+3x+y+z=6,$$
$$w+x+3y+z=44,$$
$$w+x+y+3z=2.$$

What is the value of $wxyz$?

Exercise 4:

Let x, y, and z be positive real numbers satisfying the simultaneous equations

$$x(y^2 + yz + z^2) = 3y + 10z,$$
$$y(z^2 + zx + x^2) = 21z + 24x,$$
$$z(x^2 + xy + y^2) = 7x + 28y.$$

Find $xy + yz + zx$.

Exercise 5:

Suppose $x,\ y$, and z are integers that satisfy the system of equations

$$x^2y + y^2z + z^2x = 2186,$$
$$xy^2 + yz^2 + zx^2 = 2188.$$

Find

$$x^2 + y^2 + z^2.$$

Exercise 6:

Suppose $a,\ b$, and c are real numbers that satisfy

$$a + b + c = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 5, \text{ and } abc = 1.$$

Find the value of $a^3 + b^3 + c^3$.

HOMEWORK

Problem Set I

Problem 1: 1998 AIME #3

The graph of

$$y^2 + 2xy + 40|x| = 400$$

partitions the plane into several regions. What is the area of the bounded region?

Problem 2: 1993AIME #3

The table below displays some of the results of last summer's Frostbite Falls Fishing Festival, showing how many contestants caught n, fish for various values of n.

n	0	1	2	3	...	13	14	15
number of contestants who caught n fish	9	5	7	23	...	5	2	1

(a) the winner caught 15 fish;

(b) those who caught 3 or more fish averaged 6 fish each;

(c) those who caught 12 or fewer fish averaged 5 fish each.

What was the total number of fish caught during the festival?

Problem 3: 1987 AIME #4

Find the area of the region enclosed by the graph of

$$|x-60| + |y| = \left|\frac{x}{4}\right|.$$

Problem 4: **1983 AIME #5**

Suppose that the sum of the squares of two complex numbers x and y is 7 and the sum of the cubes is 10. What is the largest real value that $x + y$ can have?

Problem 5: 1986 AIME #4

Determine $3x_4 + 2x_5$ if x_1, x_2, x_3, x_4, and x_5 satisfy the system of equations below.

$$\begin{aligned} 2x_1 + x_2 + x_3 + x_4 + x_5 &= 6 \\ x_1 + 2x_2 + x_3 + x_4 + x_5 &= 12 \\ x_1 + x_2 + 2x_3 + x_4 + x_5 &= 24 \\ x_1 + x_2 + x_3 + 2x_4 + x_5 &= 48 \\ x_1 + x_2 + x_3 + x_4 + 2x_5 &= 96 \end{aligned}$$

Problem 6: 2004 AIME II #6

Three clever monkeys divide a pile of bananas. The first monkey takes some bananas from the pile, keeps three-fourths of them, and divides the rest equally between the other two. The second monkey takes some bananas from the pile, keeps one-fourth of them, and divides the rest equally between the other two. The third monkey takes the remaining bananas from the pile, keeps one-twelfth of them, and divides the rest equally between the other two. Given that each monkey receives a whole number of bananas whenever the bananas are divided, and the numbers of bananas the first, second, and third monkeys have at the end of the process are in the ratio 3 : 2 : 1, what is the least possible total for the number of bananas?

Problem 7: **1988 AIME #6**

			*	
	74			
				186
		103		
0				

It is possible to place positive integers into the vacant twenty-one squares of the 5×5 square shown below so that the numbers in each row and column form arithmetic sequences. Find the number that must occupy the vacant square marked by the asterisk (*).

Problem 8: 1989 AIME #8

Assume that $x_1, x_2, \ldots, x_7$ are real numbers such that

$$\begin{aligned} x_1 + 4x_2 + 9x_3 + 16x_4 + 25x_5 + 36x_6 + 49x_7 &= 1, \\ 4x_1 + 9x_2 + 16x_3 + 25x_4 + 36x_5 + 49x_6 + 64x_7 &= 12, \\ 9x_1 + 16x_2 + 25x_3 + 36x_4 + 49x_5 + 64x_6 + 81x_7 &= 123. \end{aligned}$$

Find the value of

$$16x_1 + 25x_2 + 36x_3 + 49x_4 + 64x_5 + 81x_6 + 100x_7.$$

Problem 9: 2000 AIME I #7

Suppose that x, y, and z are three positive numbers that satisfy the equations

$$xyz = 1, \quad x + \frac{1}{z} = 5, \quad \text{and} \quad y + \frac{1}{x} = 29.$$

Then

$$z + \frac{1}{y} = \frac{m}{n},$$

where m and n are relatively prime positive integers. Find $m+n$.

Problem 10: 1987 AIME #10

Al walks down to the bottom of an escalator that is moving up and he counts 150 steps. His friend, Bob, walks up to the top of the escalator and counts 75 steps. If Al's speed of walking (in steps per unit time) is three times Bob's walking speed, how many steps are visible on the escalator at a given time? (Assume that this value is constant.)

Problem 11: 1988 AIME #14

Let C be the graph of $xy=1$, and denote by C^* the reflection of C in the line $y=2x$. Let the equation of C^* be written in the form

$$12x^2+bxy+cy^2+d=0.$$

Find the product bc.

Problem 12: 1984 AIME #15

Determine $x^2+y^2+z^2+w^2$ if

$$\begin{aligned}
\frac{x^2}{2^2-1}+\frac{y^2}{2^2-3^2}+\frac{z^2}{2^2-5^2}+\frac{w^2}{2^2-7^2}&=1\\
\frac{x^2}{4^2-1}+\frac{y^2}{4^2-3^2}+\frac{z^2}{4^2-5^2}+\frac{w^2}{4^2-7^2}&=1\\
\frac{x^2}{6^2-1}+\frac{y^2}{6^2-3^2}+\frac{z^2}{6^2-5^2}+\frac{w^2}{6^2-7^2}&=1\\
\frac{x^2}{8^2-1}+\frac{y^2}{8^2-3^2}+\frac{z^2}{8^2-5^2}+\frac{w^2}{8^2-7^2}&=1
\end{aligned}$$

Problem 13: **2006 AIME I #15**

Given that a sequence satisfies $x_0 = 0$ and

$$|x_k| = |x_{k-1} + 3|$$

for all integers $k \geq 1$, find the minimum possible value of

$$|x_1 + x_2 + \cdots + x_{2006}|.$$

Problem Set II

Problem 1.

Let x, y, and z be real numbers satisfying the equations

$$(x+y)(x+y+z)=18,$$
$$(y+z)(x+y+z)=30,$$
$$(z+x)(x+y+z)=24.$$

Find the sum of all possible values of $x^2+y^2+z^2$.

Problem 2.

Let a, b, and c be real numbers such that

$$a^2+b^2+c^2=1998 \text{ and } (a+b)^2+(b+c)^2+(c+a)^2=2023.$$

Find the largest possible value of $a+b+c$.

Problem 3.

Let a, b, c, and d be positive real numbers such that

$$a^2+b^2=c^2+d^2=2023,$$
$$ac=bd=1000.$$

If $S=a+b+c+d$, compute the value of $\lfloor S \rfloor$.

Problem 4:

Find x^6+y^6 if the real numbers x and y satisfy the equations

$$xy=3$$

and

$$\frac{x}{x+y^2}+\frac{y}{y+x^2}=\frac{6}{7}.$$

Problem 5:

Let $x,\ y,$ and z are real numbers such that

$$xy+\frac{7}{z}=yz+\frac{1}{x}=zx+\frac{2}{y}=\frac{1}{x+y+z}.$$

Then $(x+y+z)^6$ can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problem 6:

Let $x,\ y,$ and z be real numbers such that

$$\left(x+\frac{1}{y}\right)\left(y+\frac{1}{z}\right)\left(z+\frac{1}{x}\right)=\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right)\left(1+\frac{1}{z}\right)$$

and

$$xyz=-999.$$

Find the value of $x+y+z$.

Problem 7:

If $x,\ y,$ and z are real numbers such that

$$\frac{xy}{x+y}=4,\quad \frac{yz}{y+z}=5,\quad \text{and} \quad \frac{zx}{z+x}=6,$$

then

$$\frac{xyz}{xy+yz+zx}$$

can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problem 8:

Find $x^3 + y^3 + z^3$ if the real numbers $x,\ y$, and z satisfy the equations

$$x + y + z = 9$$

and

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{9}\,.$$

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