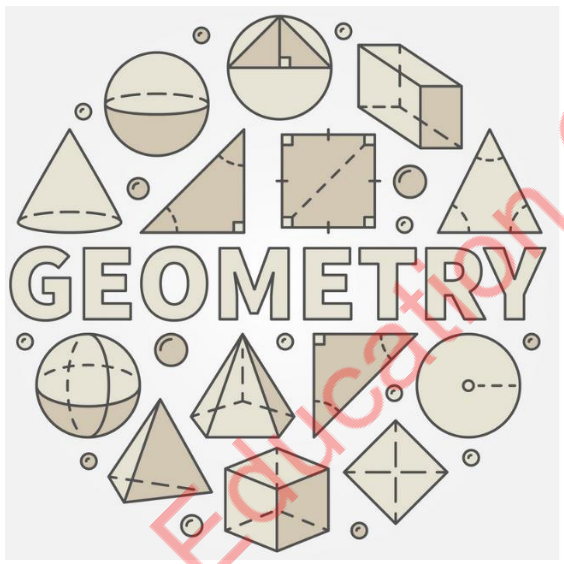
AMC 10/12 Prep Course

(Tutorial Handout Sample)



***Topic*:**

Angle Bisectors

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1. Introduction

In geometry, the **angle bisector theorem** is concerned with the relative lengths of the two segments that a triangle's side is divided into by a line that bisects the opposite angle. It equates their relative lengths to the relative lengths of the other two sides of the triangle.

The angle bisector theorem frequently show up as an important intermediate step in problems involving the measurements of triangles. The theorem can be utilized to quickly and efficiently solve many difficult geometry problems on the AMC 10/12, AIME, and Olympiads.

Important Strategy:

- **Keyword Based Indication**: If the keyword phrase "*angle bisector*" or "*bisects the angle*" is presented in a problem, you can definitely use the **Angle Bisector Theorem**.

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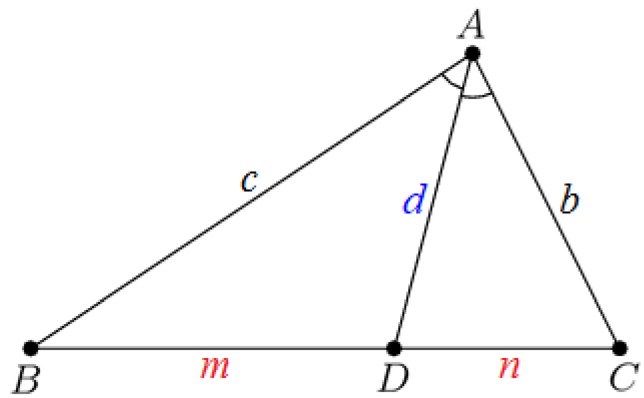
2. Fundamental Results



To bisect an angle means to cut it into two equal parts or angles. Angle bisectors in a triangle have a characteristic property of dividing the opposite side in the ratio of the adjacent sides.

Result 1: Angle Bisector Theorem

Let AD be the bisector of $\angle A$ with D on BC in ΔABC. If $b = AC, c = AB, m = BD$, and $n = CD$, then

$$\frac{c}{m} = \frac{b}{n}.$$

Likewise, the converse of this theorem holds as well.

Result 2: The Formula for the Length of an Angle Bisector

Let AD be the angle bisector of $\angle A$ with D on BC in ΔABC. Let $d = AD, m = BD$, and $n = CD$. Then

$$d^2 = bc - mn = \frac{bc}{(b+c)^2}\left((b+c)^2 - a^2\right),$$

where a, b, and c are the sides opposite A, B, and C, respectively.

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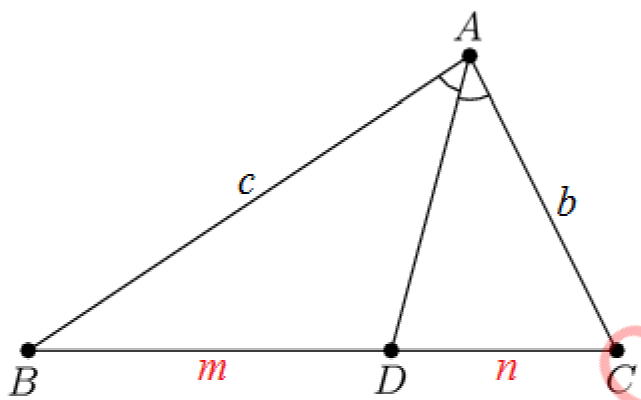
3. Proofs of Angle Bisector Theorem

Angle Bisector Theorem



The Angle Bisector Theorem states that given triangle ΔABC and angle bisector AD, where D is on side BC,

$$\frac{c}{m} = \frac{b}{n}.$$

Likewise, the converse of this theorem holds as well.

We will develop six methods to prove this important theorem.

Method 1

Construct a line parallel to AD and ray BA to intersect at E, as shown below.

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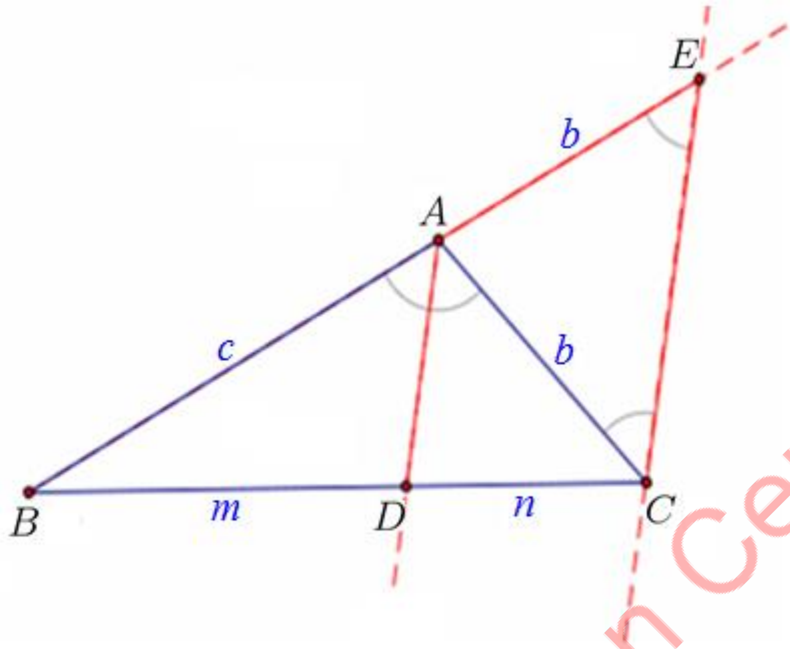

Note that $\angle BAD = \angle DAC$ (bisected angle), $\angle BAD = \angle AEC$ (corresponding angles), and $\angle DAC = \angle ACE$ (alternate interior angles). Thus,

$$\angle AEC = \angle ACE,$$

which implies that ΔACE is isosceles and

$$AE = AC = b.$$

By the side-splitter theorem, we get

$$\frac{AB}{AE} = \frac{BD}{CD}.$$

Since $AE = AC$, it follows that

$$\frac{c}{b} = \frac{m}{n}.$$

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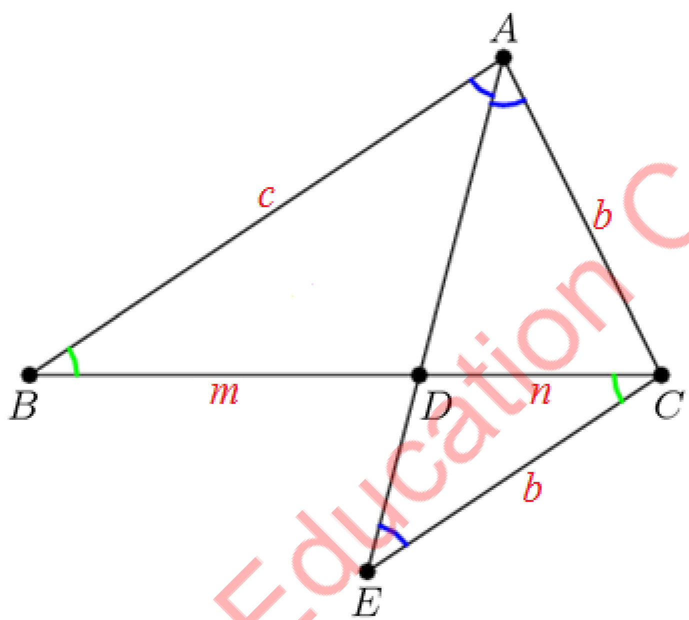
Method 2

Because of the ratios and equal angles in the theorem, we think of similar triangles. There are not any similar triangles in the figure as it now stands, however. So, we think to draw in a carefully chosen line or two. Extending AD until it hits the line through C parallel to AB does just the trick, as shown below:



Since $AB \parallel CE$, it follows that

$$\angle BAE = \angle CEA \quad \text{and} \quad \angle BCE = \angle ABC.$$

Now we have:

$$\angle CEA = \angle BAE = \angle CAE.$$

Thus, ΔACE is isosceles, with

$$AC = CE = b.$$

By the AA similarity postulate,

$$\Delta DAB \cong \Delta DEC.$$

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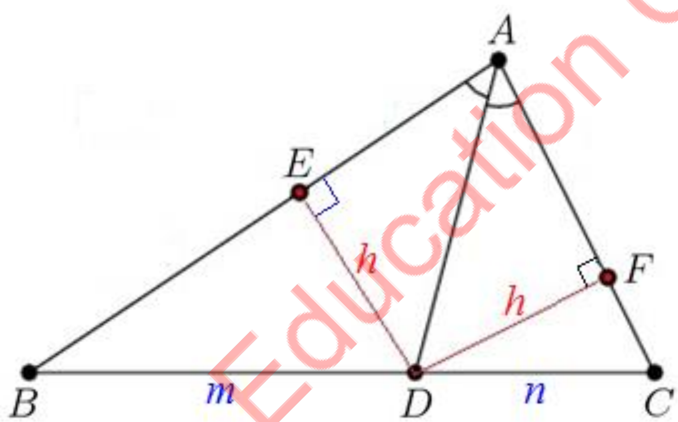
By the properties of similar triangles, we arrive at our desired result:

$$\frac{c}{m} = \frac{b}{n}.$$

Method 3

Since ΔABD and ΔACD have the same altitude, it follows that

$$\frac{\text{Area}(\Delta ABD)}{\text{Area}(\Delta ACD)} = \frac{m}{n}.$$



Now let E and F be the feet of the perpendiculars from D to AB and AC, respectively. Recall that every point on the angle bisector of an angle is equidistant to the sides of the angle. So the height h to AB is equal to the height to AC. Thus

$$\frac{\text{Area}(\Delta ABD)}{\text{Area}(\Delta ACD)} = \frac{\frac{ch}{2}}{\frac{bh}{2}} = \frac{c}{b}.$$

Hence

$$\frac{c}{b} = \frac{m}{n},$$

or

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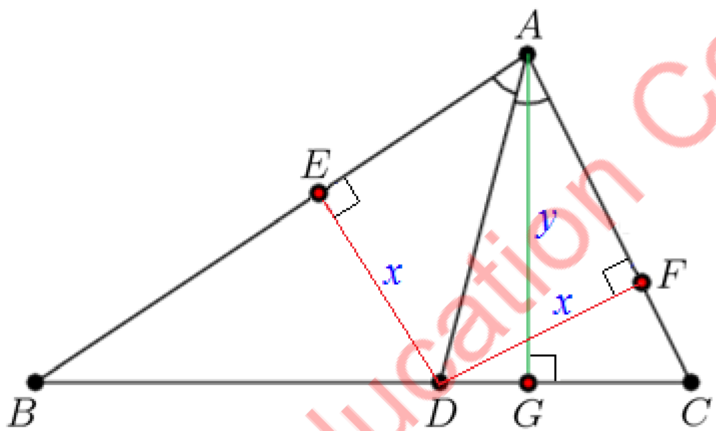
$$\frac{c}{m} = \frac{b}{n}.$$

We can prove the converse by the Phantom Point Method, since we can find m and n in terms of a, b, and c, and prove that the points are the same.

Method 4



Let E and F be the feet of the perpendiculars from D to AB and AC, respectively. Let G be the foot of perpendicular from A to BC. In addition, let $DE = x$ and $AG = y$. Note that

$$\angle ADE = \angle ADF.$$

According to the ASA congruence rule, right triangles ADE and ADF are congruent. Thus,

$$DE = DF = x.$$

By finding the area of a right triangle in two different ways, we have

$$\text{Area}(\Delta ABD) = \frac{1}{2}cx = \frac{1}{2}my,$$

and

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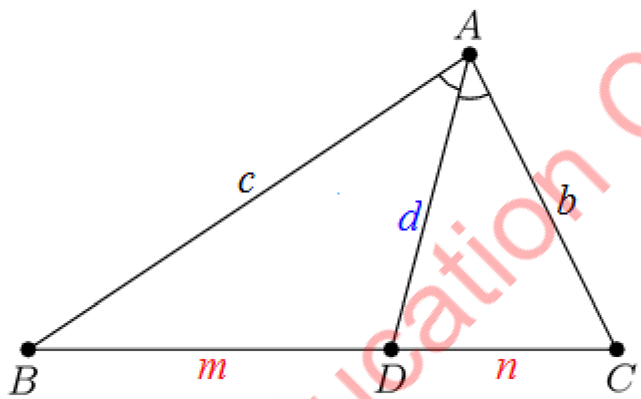
$$\text{Area}(\Delta ACD) = \frac{1}{2}bx = \frac{1}{2}ny.$$

Hence,

$$\frac{y}{x} = \frac{c}{m} = \frac{b}{n}.$$

Method 5



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Let $AD = d$. Using the side-angle-side method, we can express the area of ΔABD in two ways:

$$\text{Area}(\Delta ABD) = \frac{1}{2}cd\sin\angle BAD = \frac{1}{2}md\sin\angle ADB,$$

which implies that

$$\frac{\sin\angle ADB}{\sin\angle BAD} = \frac{c}{m}.$$

Likewise, ΔACD can be expressed in two different ways:

$$\text{Area}(\Delta ACD) = \frac{1}{2}bd\sin\angle CAD = \frac{1}{2}nd\sin\angle ADC,$$

$$\frac{\sin\angle ADC}{\sin\angle CAD} = \frac{b}{n}.$$

But $\angle CAD = \angle BAD$ and

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$$\sin\angle ADC = \sin\angle ADB$$

since $\angle ADC = 180° - \angle ADB$.

Therefore, we can substitute back into our previous equation to get

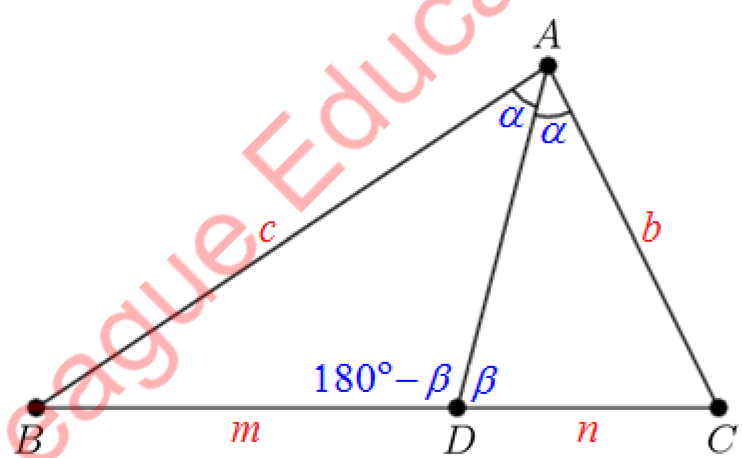
$$\frac{\sin\angle ADB}{\sin\angle BAD} = \frac{b}{n}.$$

We conclude that

$$\frac{\sin\angle ADB}{\sin\angle BAD} = \frac{c}{m} = \frac{b}{n}.$$

In both cases, if we reverse all the steps, we see that everything still holds and thus the converse holds.

Method 6: Trigonometric Approach



Using the Law of Sines for ΔABD and ΔACD, we have

$$\frac{\sin\alpha}{m} = \frac{\sin(180^o - \beta)}{c}$$

and

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$$\frac{\sin\alpha}{n} = \frac{\sin\beta}{b}.$$

Note that $\sin(180^{o} - \beta) = \sin\beta$. We obtain

$$\frac{c}{m} = \frac{b}{n}.$$

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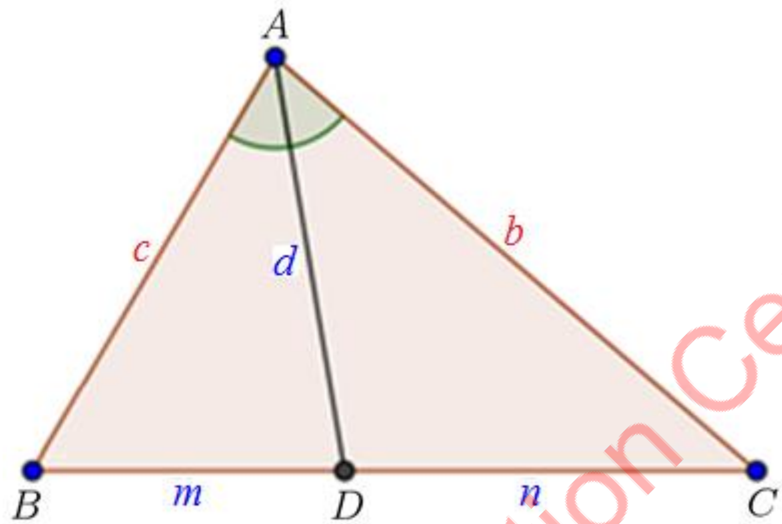
4. Proofs of the Formula for the Length of An Angle Bisector

Theorem:



Let AD be the angle bisector of $\angle A$ with D on BC in ΔABC. Let $d = AD, m = BD$, and $n = CD$. Then

$$d^2 = bc - mn = \frac{bc}{(b+c)^2}((b+c)^2 - a^2),$$

where a, b, and c are the sides opposite A, B, and C, respectively.

We will give three proofs to this theorem.

First Proof:

Draw the circumcircle of ΔABC, extend AD to intersect the circle at E, and connect CE.

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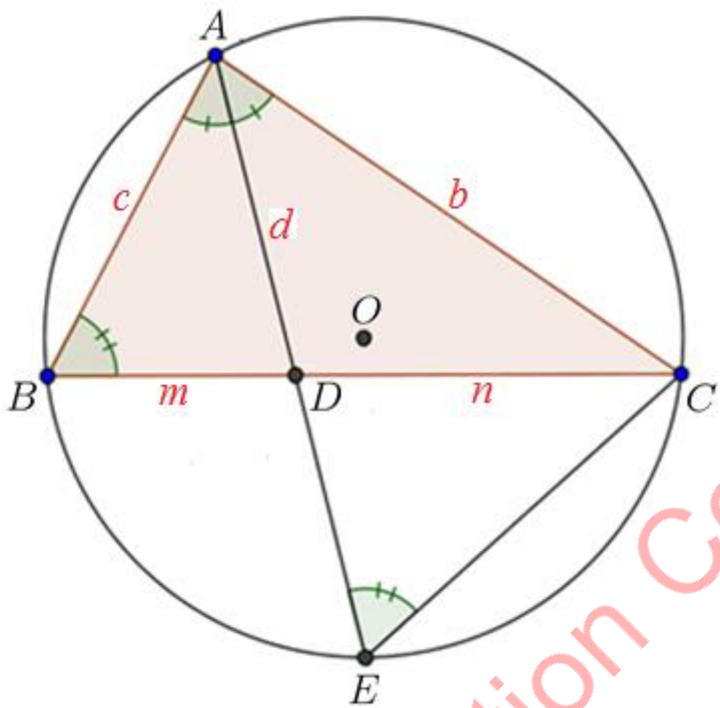

Recall that in a circle, two inscribed angles with the same intercepted arc are congruent. Thus,

$$\angle ABD = \angle AEC.$$

Note that $\angle BAD = \angle EAC$. By the angle-angle similarity criterion, we deduce that

$$\Delta ABD \sim \Delta AEC.$$

Thus,

$$\frac{AE}{c} = \frac{b}{d},$$

or

$$DE + d = AF = \frac{bc}{d},$$

which implies that

$$DF = \frac{bc - d^2}{d}.$$

Using the Intersecting Chord (Power of a Point) Theorem gives

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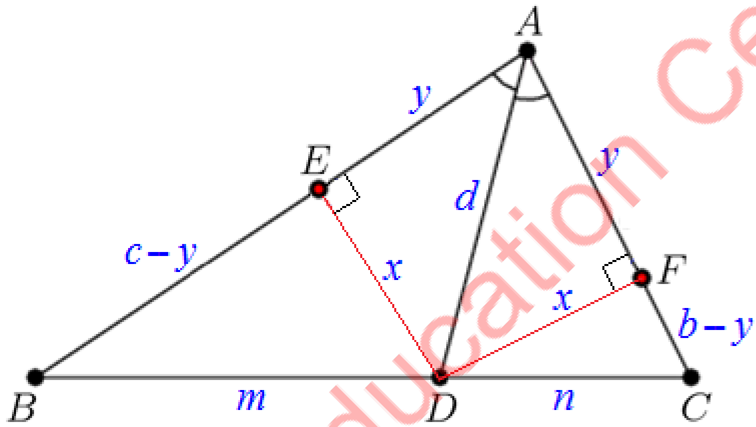
$$mn = d \cdot DE = d \cdot \frac{bc - d^2}{d} = bc - d^2.$$

Hence,

$$d^2 = bc - mn.$$

Second Proof:



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Let E and F be the feet of the perpendiculars from D to AB and AC, respectively. Also, let $DE = x$ and $AE = y$. Then

$$BE = c - y.$$

Note that $\angle ADF = \angle ADE$. According to the ASA congruence rule,

$$\Delta ADF \cong \Delta ADE,$$

so

$$DF = DE = x, \;\; AF = AE = y, \;\; \text{and} \quad CF = b - y.$$

Applying the Pythagorean Theorem to $\Delta ADE,\ \Delta BDE$, and ΔCDF, respectively, gives:

$$d^2 = x^2 + y^2, \quad (1)$$

$$m^2 = x^2 + (c - y)^2, \quad (2)$$

$$n^2 = x^2 + (b - y)^2. \quad (3)$$

Subtracting the first equation from the second yields:

$$m^2 - d^2 = -2cy + c^2,$$

or

$$m^2 = d^2 - 2cy + c^2. \quad (4)$$

Similarly, we have:

$$n^2 = d^2 - 2by + b^2. \quad (5)$$

Multiplying Equation (4) by b, multiplying Equation (5) by $-c$, and then adding the two equations, we obtain:

$$m^2b - n^2c = d^2b - d^2c + bc^2 - cb^2 = (b - c)(d^2 - bc).$$

Using the angle bisector theorem gives:

$$bm = cn.$$

Thus,

$$m^2b - n^2c = m(bm) - n(cn) = m(cn) - n(bm) = (b - c)mn,$$

which implies that

$$(b - c)mn = (b - c)(d^2 - bc).$$

Hence,

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$$d^2 = bc - mn.$$

Note that

$$\frac{m}{n} = \frac{c}{b},$$

yielding

$$\frac{a}{n} = \frac{m+n}{n} = \frac{c+b}{b}.$$

Thus,

$$n = \frac{ab}{b+c}.$$

Similarly, or by symmetry, we have:

$$m = \frac{ac}{b+c}.$$

Hence,

$$d^2 = bc - mn = bc - \frac{ac}{b+c} \cdot \frac{ab}{b+c} = \frac{bc}{(b+c)^2}((b+c)^2 - a^2).$$

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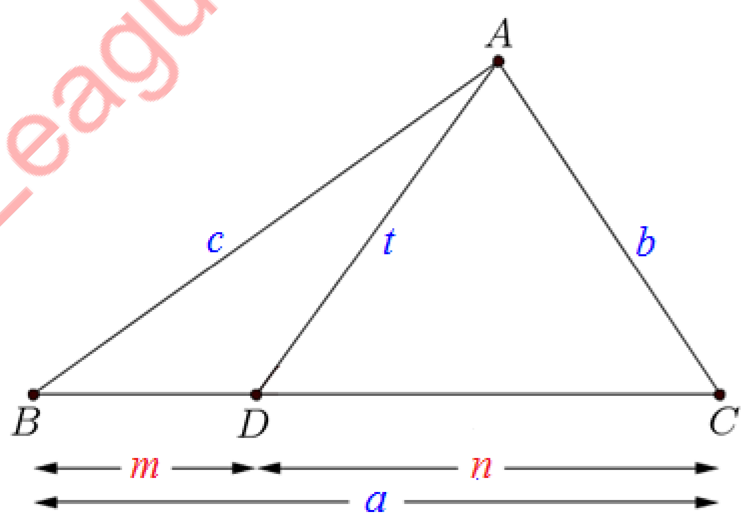
Third Proof:

We use Stewart's Theorem to find the length of an angle bisector.



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Recall that Stewart's Theorem states that in ΔABC with sides $a, b,$ and c, if D is a point on BC such that $BD = m, DC = n,$ and $AD = t$, then

$$t^2 = \frac{b^2m + c^2n}{m+n} - mn.$$

If AD is the angle bisector of $\angle A$, then by Stewart's Theorem,

$$AD^2 = \frac{b^2m + c^2n}{m+n} - mn.$$

By the angle bisector theorem,

$$bm = cn.$$

Therefore,

$$AD^2 = \frac{b(bm) + c(cn)}{m+n} - mn = \frac{b(cn) + c(bm)}{m+n} - mn = bc - mn.$$

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5. Problem Solving

Example 1. 2022 AMC 10A #13

Let ΔABC be a scalene triangle. Point P lies on $\overline{BC}$ so that $\overline{AP}$ bisects $\angle BAC$. The line through B perpendicular to $\overline{AP}$ intersects the line through A parallel to $\overline{BC}$ at point D. Suppose

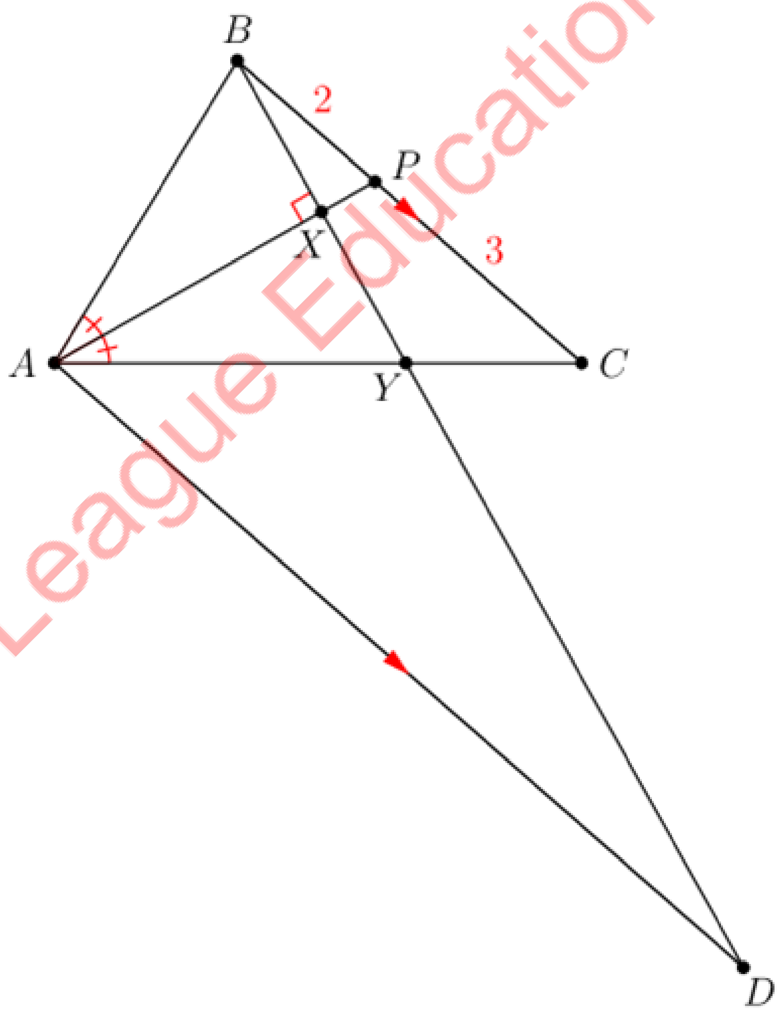
$BP = 2$ and $PC = 3$.

What is AD?

(A) 8 **(B)** 9 **(C)** 10 **(D)** 11 **(E)** 12

Answer: **(C)**

Solution:



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Let $\overline{BD}$ intersect $\overline{AP}$ and $\overline{AC}$ at X and Y, respectively. By the ASA congruence postulate,

$$\Delta ABX \cong \Delta AYX.$$

Now let $AB = AY = 2t$. Then by the Angle Bisector Theorem,

$$AC = 3t,$$

so

$$CY = t.$$

By the Alternate Interior Angles Theorem,

$$\angle YAD = \angle YCB \quad \text{and} \quad \angle YDA = \angle YBC.$$

By the AA similarity criteria,

$$\Delta ADY \sim \Delta CBY$$

with the ratio of similitude

$$\frac{AY}{CY} = 2.$$

Hence,

$$AD = 2CB = 2(BP + PC) = 2(2 + 3) = 10.$$

Example 2: 2016 MathCounts National Sprint #30

In isosceles triangle ABC with base BC of length 23 cm, points P and Q are chosen on side BC with $BP = QC = 9$ cm. If segments AP and AQ trisect angle BAC, what is the perimeter of triangle ABC?

Answer: 50

Solution 1:

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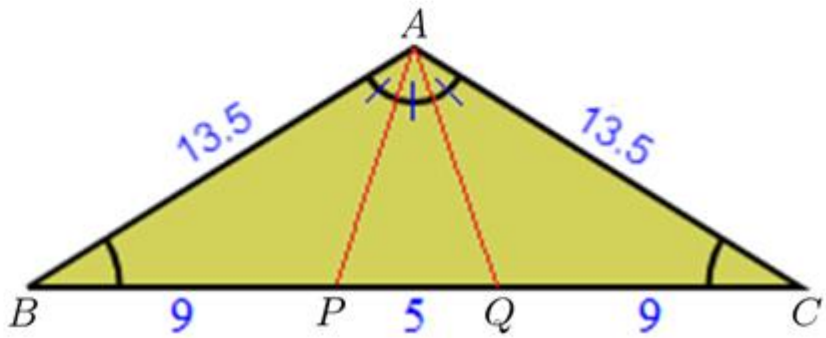

Note that

$$PQ = BC - BP - QC = 23 - 9 - 9 = 5.$$

By the angle-bisector theorem,

$$\frac{AB}{AQ} = \frac{9}{5}.$$

Now let

$$AB = 9x \quad \text{and} \quad AQ = 5x.$$

By symmetry,

$$AP = AQ = 5x.$$

Using the angle bisector length formula,

$$AP^2 = AB \cdot AQ - BP \cdot PQ.$$

Thus,

$$(5x)^2 = 9x \cdot 5x - 9 \cdot 5,$$

which implies that

$$x = \frac{3}{2}.$$

Hence, the perimeter of triangle ABC is:

$$18x + 23 = 18 \times \frac{3}{2} + 23 = 50.$$

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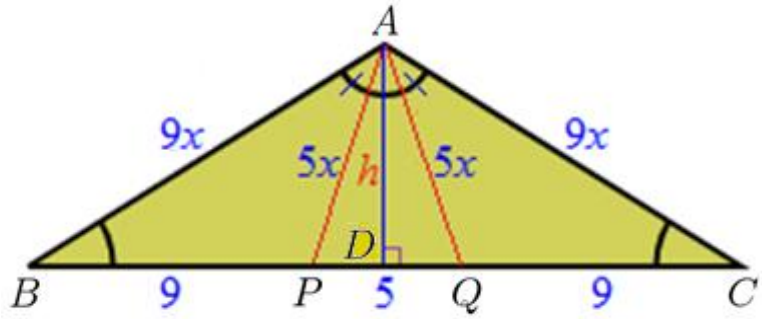
Solution 2:



Let D be the foot of the perpendicular from A to BC, and $AD = h$. Then

$$PD = \frac{PQ}{2} = \frac{5}{2}.$$

Applying the Pythagorean Theorem to ΔABD and ΔAPD, we have:

$$h^2 = (9x)^2 - \left(\frac{23}{2}\right)^2,$$

$$h^2 = (5x)^2 - \left(\frac{5}{2}\right)^2.$$

Thus,

$$(9x)^2 - \left(\frac{23}{2}\right)^2 = (5x)^2 - \left(\frac{5}{2}\right)^2,$$

which implies that

$$x = \frac{3}{2}.$$

Therefore, the perimeter of triangle ABC is:

$$18x + 23 = 18 \times \frac{3}{2} + 23 = 50.$$

Example 3: 2018 AMC 10A #24

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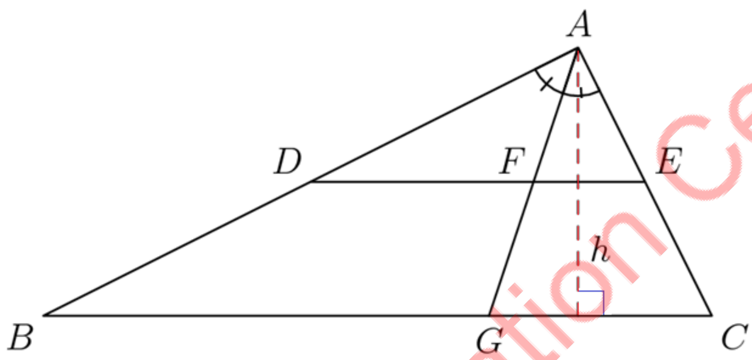
Triangle ABC with $AB = 50$ and $AC = 10$ has area 120. Let D be the midpoint of $\overline{AB}$, and let E be the midpoint of $\overline{AC}$. The angle bisector of $\angle BAC$ intersects $\overline{DE}$ and $\overline{BC}$ at F and G, respectively. What is the area of quadrilateral $FDBG$?

(A) 60 **(B)** 65 **(C)** 70 **(D)** 75 **(E)** 80

Answer: **(D)**

Solution 1



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Let a be the height of ΔABC and $BC = a$. Then

$$\text{Area}(\Delta ABC) = \frac{ah}{2} = 120,$$

which implies that

$$ah = 240.$$

Using the angle bisector theorem,

$$\frac{BG}{AB} = \frac{CG}{AC},$$

which gives:

$$\frac{BG}{CG} = \frac{AB}{AC} = \frac{50}{10} = 5.$$

Thus,

$$BG = \frac{5}{5+1}BC = \frac{5a}{6}.$$

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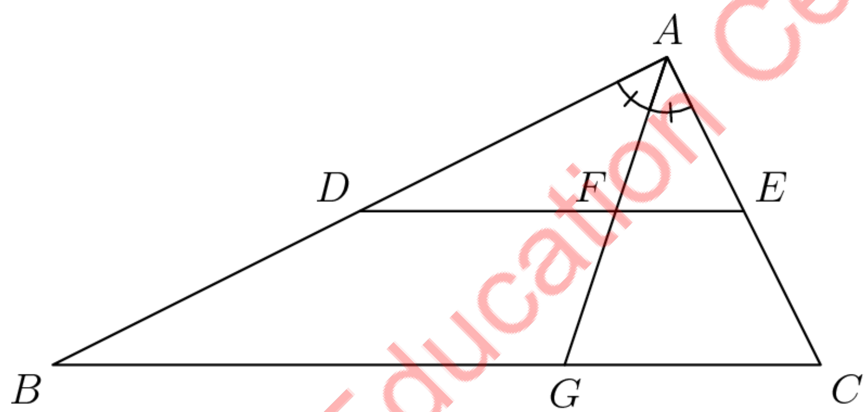
Since D and E are, respectively, the midpoints of $\overline{AB}$ and $\overline{AC}$, it follows that F is the midpoint of $\overline{AG}$. Thus, ΔABG and ΔADF are similar with a scale factor of 2. So

$$DF = \frac{BG}{2} = \frac{5a}{12}.$$

Hence, the area of trapezoid $FDBG$ with upper base DF, lower base BG, and height $\frac{h}{2}$ is:

$$\frac{DF + BG}{2} \cdot \frac{h}{2} = \frac{\frac{5a}{12} + \frac{5a}{6}}{2} \cdot \frac{h}{2} = \frac{5ah}{16} = \frac{5 \times 240}{16} = 75\,.$$

Solution 2



Since the angle bisector AG splits the segment in the same ratio as $\frac{AB}{AC}$, it follows that:

$$\frac{BG}{CG} = \frac{AB}{AC} = \frac{50}{10} = 5.$$

Recall that triangles of equal heights have areas propositional to their corresponding bases.

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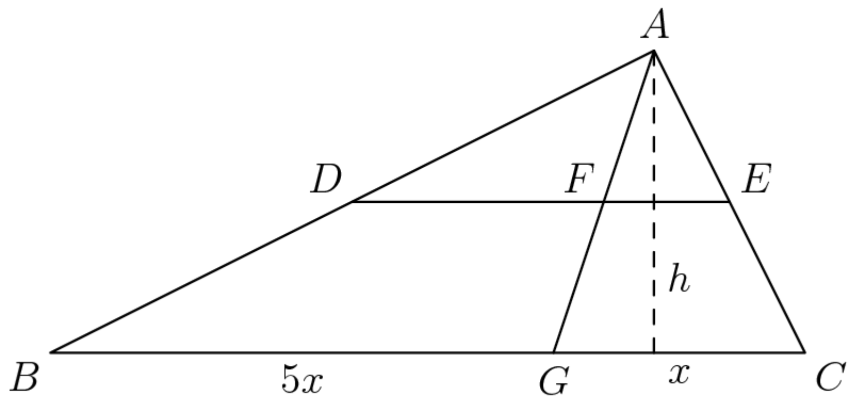

Thus,

$$\frac{\text{Area}(\Delta ABG)}{\text{Area}(\Delta ACG)} = \frac{BG}{CG} = 5,$$

which implies that

$$\text{Area}(\Delta ABG) = \frac{5}{5+1}\text{Area}(\Delta ABC) = \frac{5}{6} \times 120 = 100.$$

Since DF is midsegment of ΔABG, it follows that ΔABG and ΔADF are similar with a scale factor of 2, and so

$$\frac{\text{Area}(\Delta ABG)}{\text{Area}(\Delta ADF)} = 2^2 = 4,$$

which implies that

$$\text{Area}(\Delta ADF) = \frac{\text{Area}(\Delta ABG)}{4} = \frac{100}{4} = 25.$$

Hence,

$$\text{Area}(\text{Quadrilateral } FDBG) = \text{Area}(\Delta ABG) - \text{Area}(\Delta ADF) = 100 - 25 = 75.$$

Solution 3

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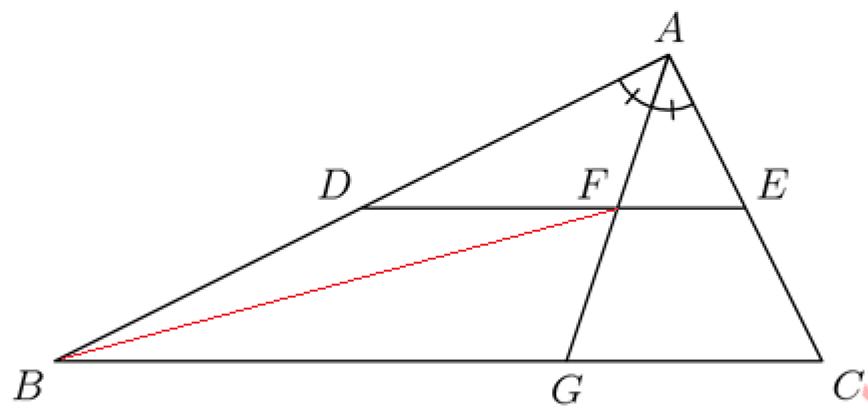

Connect BF. Using the angle bisector theorem, we have

$$\frac{BG}{CG} = \frac{AB}{AC} = \frac{50}{10} = 5.$$

Recall that triangles of equal heights have areas propositional to their corresponding bases. Thus,

$$\frac{\text{Area}(\Delta ABG)}{\text{Area}(\Delta ACG)} = \frac{BG}{CG} = 5,$$

which implies that

$$\text{Area}(\Delta ABG) = \frac{5}{5+1}\text{Area}(\Delta ABC) = \frac{5}{6} \times 120 = 100.$$

Since F is the midpoint of AG, we have:

$$\frac{\text{Area}(\Delta BGF)}{\text{Area}(\Delta BAF)} = \frac{GF}{AF} = 1,$$

which yields:

$$\text{Area}(\Delta BGF) = \text{Area}(\Delta BAF) = \frac{\text{Area}(\Delta ABG)}{2} = \frac{100}{2} = 50.$$

Similarly,

$$\text{Area}(\Delta FBD) = \text{Area}(\Delta FAD) = \frac{\text{Area}(\Delta BAF)}{2} = \frac{50}{2} = 25.$$

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Hence,

$$\text{Area}(\text{Quadrilateral } FDBG) = \text{Area}(\Delta BGF) + \text{Area}(\Delta FBD) = 50 + 25 = 75.$$

Solution 4

Note that DE is midsegment of ΔABC. Thus, ΔABC and ΔADE are similar with a scale factor of 2. So

$$\frac{\text{Area}(\Delta ABC)}{\text{Area}(\Delta ADE)} = 2^2 = 4,$$

which implies that

$$\text{Area}(\Delta ADE) = \frac{\text{Area}(\Delta ABC)}{4} = \frac{120}{4} = 30.$$

Subsequently,

$$\text{Area}(\text{Quadrilateral } EDBC) = \text{Area}(\Delta ABC) - \text{Area}(\Delta ADE) = 120 - 30 = 90.$$

Example 4: 2000 AMC 12 #19

In triangle ABC, $AB = 13,\ BC = 14,\ AC = 15$. Let D denote the midpoint of $\overline{BC}$ and let E denote the intersection of $\overline{BC}$ with the bisector of angle BAC. Which of the following is closest to the area of the triangle ADE?

(A) 2 (B) 2.5 (C) 3 (D) 3.5 (E) 4

Answer: (C)

Solution

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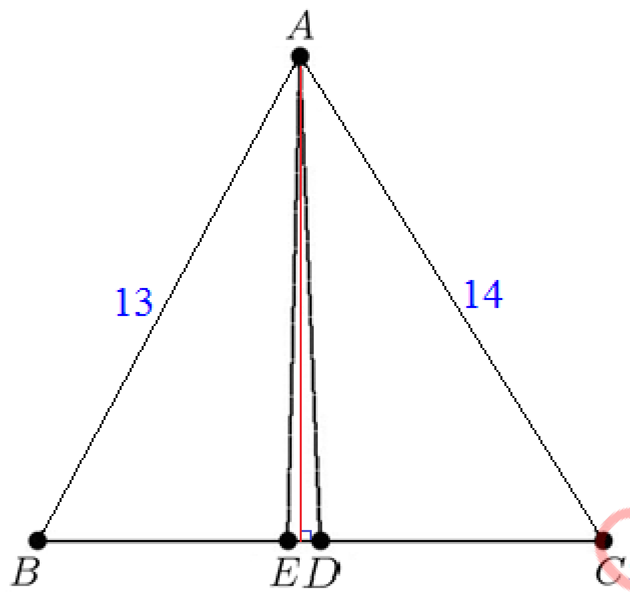

Dropping an altitude from A to BC, we can split the large triangle into a 5-12-13 and a 9-12-15 triangle. So the altitude has length 12.

By the Angle Bisector Theorem,

$$\frac{13}{BE} = \frac{15}{14 - BE},$$

which implies that

$$BE = \frac{13}{2}.$$

Since D is the midpoint of BC, it follows that $BD = 7$. Thus, the base of ΔADE is:

$$DE = BD - BE = 7 - \frac{13}{2} = \frac{1}{2}.$$

Hence, the area of ΔADE is:

$$\frac{\frac{1}{2} \times 12}{2} = 3.$$

Example 5: **2002 AMC 12A #23**

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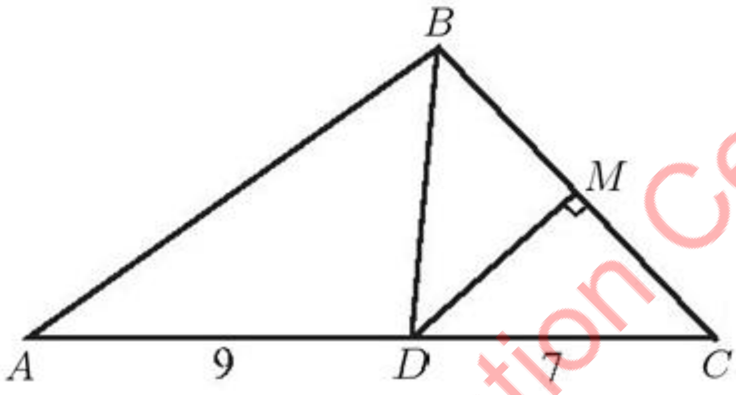
In triangle ABC, side AC and the perpendicular bisector of BC meet in point D, and BD bisects $\angle ABC$. If $AD = 9$ and $DC = 7$, what is the area of triangle ABD ?

(A) 14 (B) 21 (C) 28 (D) $14\sqrt{5}$ (E) $28\sqrt{5}$

Answer: **(D)**

Solution 1



By the angle-bisector theorem,

$$\frac{AB}{BC} = \frac{9}{7}.$$

Now let

$$AB = 9x \quad \text{and} \quad BC = 7x.$$

Let M be the midpoint of BC. Since M is on the perpendicular bisector of BC, it follows that

$$BD = DC = 7.$$

Using the formula for the length of an angle bisector, we have

$$BD^2 = AB \cdot BC - AD \cdot DC.$$

Thus,

$$7^2 = 9x \cdot 7x - 9 \cdot 7,$$

which implies that

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$$x = \frac{4}{3}.$$

So

$$AB = 9 \cdot \frac{4}{3} = 12.$$

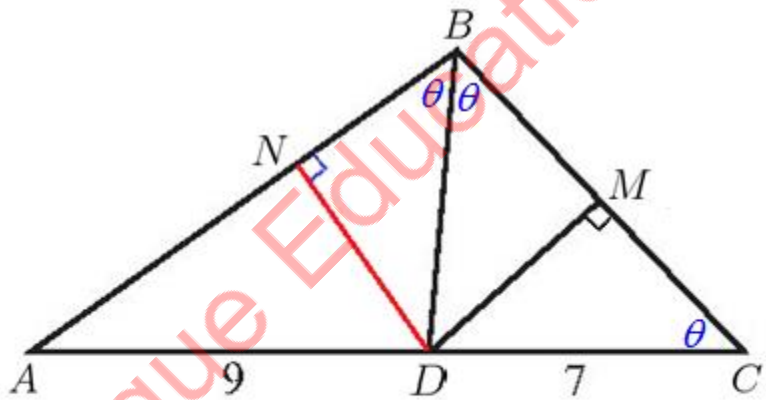
The semiperimeter of ΔABD is:

$$\frac{12 + 7 + 9}{2} = 14.$$

Apply Heron's formula to obtain the area of ΔABD with sides 12, 7, and 9 as

$$\sqrt{14 \cdot (14 - 12) \cdot (14 - 7) \cdot (14 - 9)} = \sqrt{14 \cdot 2 \cdot 5 \cdot 7} = 14\sqrt{5}.$$

Solution 2



Let M be the midpoint of BC. Connect DM. Since M is on the perpendicular bisector of BC, it follows that

$$DB = DC = 7, \quad \text{and} \quad \angle DBC = \angle DCB.$$

Note that $\angle DBC = \angle ABD$. Thus,

$$\angle ABD = \angle DCB.$$

By the AA Similarity Postulate,

$$\Delta ABC \sim \Delta ADB,$$

which implies that

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$$\frac{AC}{AB} = \frac{AB}{AD} = \frac{BC}{DB},$$

or

$$\frac{16}{AB} = \frac{AB}{9} = \frac{BC}{7}.$$

It follows that

$$AB = 12, \qquad BC = \frac{28}{3}.$$

Notice that

$$BM = \frac{BC}{2} = \frac{14}{3}.$$

Applying the Pythagorean Theorem to ΔCDB gives:

$$DM = \sqrt{DB^2 - BM^2} = \sqrt{7^2 - \left(\frac{14}{3}\right)^2} = \frac{7\sqrt{5}}{3}.$$

Now let N be the foot of the perpendicular from D to AB. By the ASA congruence criterion,

$$\Delta DBN \cong \Delta DBM,$$

and so

$$DN = DM = \frac{7\sqrt{5}}{3}.$$

Hence, the area of ΔABD is:

$$\frac{AB \cdot DN}{2} = \frac{12 \cdot \frac{7\sqrt{5}}{3}}{2} = 14\sqrt{5}.$$

OR

Because ΔABD and ΔCBD have the same height, they have areas propositional to their corresponding bases:

$$\frac{\text{Area}(\Delta ABD)}{\text{Area}(\Delta CBD)} = \frac{AD}{CD} = \frac{9}{7}.$$

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Therefore,

$$\text{Area}(\Delta ABD) = \frac{9}{7}\text{Area}(\Delta CBD) = \frac{9}{7} \cdot \frac{BC \cdot DM}{2} = \frac{9}{7} \cdot \frac{\frac{28}{3} \cdot \frac{7\sqrt{5}}{3}}{2} = 14\sqrt{5}.$$

Solution 3

By the angle bisector theorem,

$$\frac{AB}{BC} = \frac{9}{7}.$$

Let $AB = 9x$ and $BC = 7x$, let

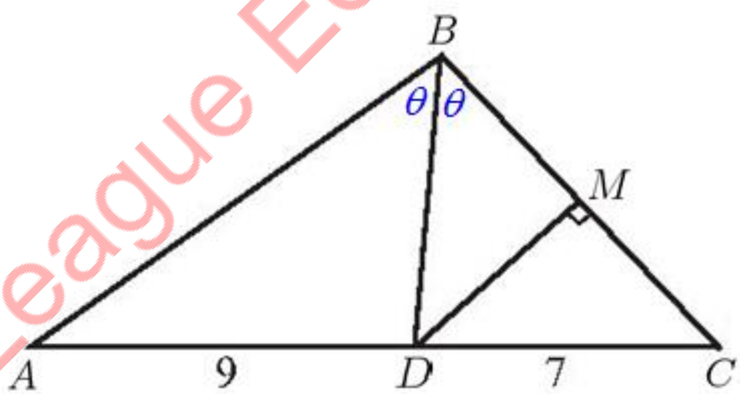
$$\angle ABD = \angle CBD = \theta\,,$$

and let M be the midpoint of BC. Since M is on the perpendicular bisector of BC, it follows that

$$BD = DC = 7.$$

Then

$$\cos\theta = \frac{\frac{7x}{2}}{7} = \frac{x}{2}.$$



Applying the Law of Cosines to ΔABD yields

$$9^2 = (9x)^2 + 7^2 - 2(9x)(7)\cos\theta,$$

from which

$$x = \frac{4}{3}$$

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and then

$$\cos\theta = \frac{x}{2} = \frac{2}{3}.$$

Recall that the measure of an exterior angle of a triangle is equal to the sum of the measures of the two non-adjacent interior angles of the triangle. So

$$\angle ADB = \angle ABD + \angle CBD = 2\theta.$$

Using the sine double-angle identity, we have:

$$\sin\angle ADB = 2\sin\theta\cos\theta = 2\left(\sqrt{1-\left(\frac{2}{3}\right)^2}\right)\left(\frac{2}{3}\right) = \frac{4\sqrt{5}}{9}.$$

Using the Side-angle-side method, we obtain the area of ΔABD as

$$\frac{1}{2}AD\cdot BD\cdot\sin\angle ADB = \frac{1}{2}\cdot 9\cdot 7\cdot\frac{4\sqrt{5}}{9} = 14\sqrt{5}.$$

Example 6: 2021 Fall AMC 12A #13

The angle bisector of the acute angle formed at the origin by the graphs of the lines $y = x$ and $y = 3x$ has equation $y = kx$. What is k?

(A) $\frac{1+\sqrt{5}}{2}$ **(B)** $\frac{1+\sqrt{7}}{2}$ **(C)** $\frac{2+\sqrt{3}}{2}$ **(D)** 2 **(E)** $\frac{2+\sqrt{5}}{2}$

Answer: (A)

Solution 1:

Consider the line $x = 1$, which intersects the lines $y = x$, $y = 3x$, and $y = kx$ at $A = (1,1)$, $B = (1,3)$, and $C = (1,k)$, respectively, as shown below.

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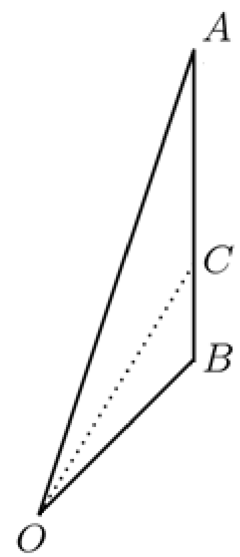

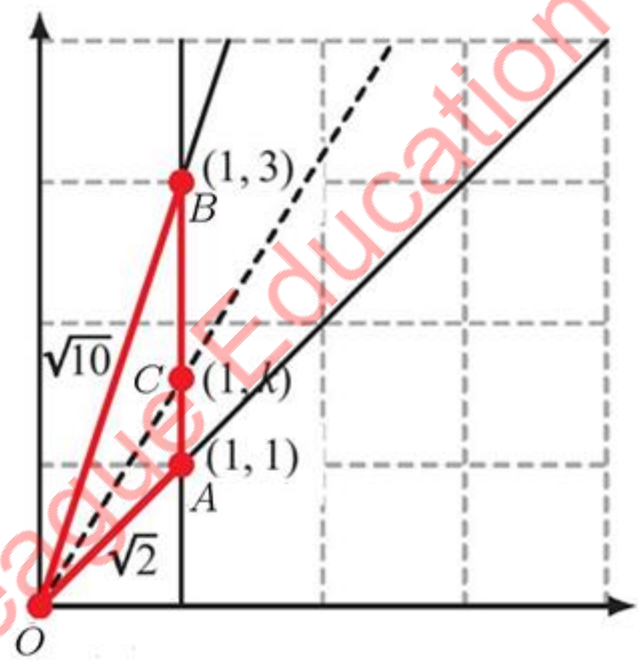

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Then

$$OA = \sqrt{2},\ \ OB = \sqrt{10},\ \ AC = k - 1,\ \ \text{and}\ \ BC = 3 - k.$$

By the angle bisector theorem,

$$\frac{BC}{AC} = \frac{OB}{OA}$$

or

$$\frac{3-k}{k-1} = \frac{\sqrt{10}}{\sqrt{2}} = \sqrt{5},$$

for which

$$k = \frac{\sqrt{5}+1}{2}.$$

Solution 2:

Consider the isosceles triangle OAB with coordinates

$$O = (0,0),\ \ A = (1,3),\ \ B = (\sqrt{5},\sqrt{5}).$$

Let M be the midpoint of AB. Then

$$M = \left(\frac{1+\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2}\right).$$

Note that A and B lie on the lines $y = 3x$ and $y = x$, respectively. Since OM is the angle bisector of $\angle AOB$, it follows that M lies on the line $y = kx$. Hence, its slope is:

$$k = \frac{\frac{3+\sqrt{5}}{2}}{\frac{1+\sqrt{5}}{2}} == \frac{\sqrt{5}+1}{2}.$$

Solution 3:

Consider points $P = (\sqrt{5},\sqrt{5})$ and $Q = (1,3)$, which lie on the lines $y = x$ and $y = 3x$, respectively. Then the line $y = kx$ is the perpendicular bisector of PQ.

Note that the slope of PQ is:

$$\frac{\sqrt{5}-3}{\sqrt{5}-1}.$$

Thus, the slope of the line $y = kx$ is:

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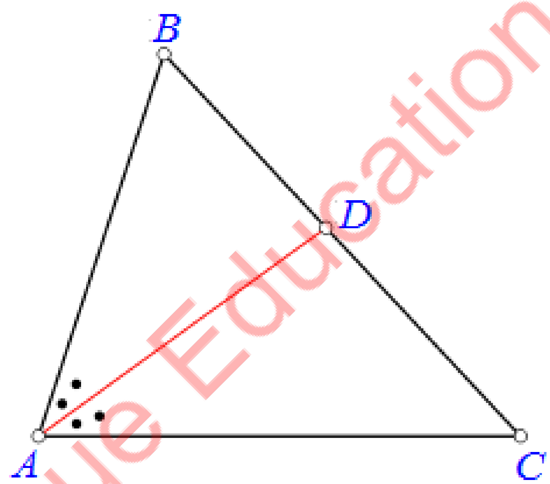
$$k = -\frac{\sqrt{5}-1}{\sqrt{5}-3} == \frac{\sqrt{5}+1}{2}.$$

Example 7: 2014 Harvard-MIT Math Tournament Geometry Test #7

In triangle ABC, $AB = 10$ and $AC = 16$. D is a point on BC such that AD bisects angle BAC, and $CD = 8$. Find the length of AD.

Answer: $2\sqrt{30}$

Solution 1:



By the angle bisector theorem,

$$\frac{AB}{BD} = \frac{AC}{CD}.$$

Note that $AB = 10,\ AC = 16,\ CD = 8$. So

$$BD = 5.$$

Using the formula for the length of an angle bisector, we have

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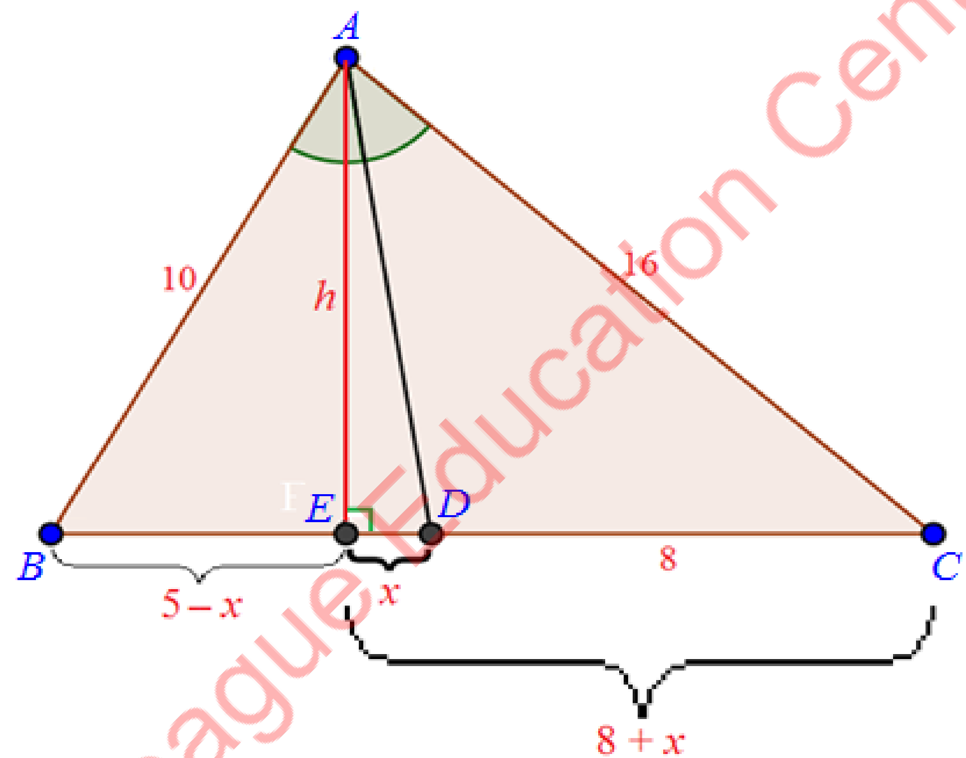
$$AD^2 = AB \cdot AC - BD \cdot CD = 10 \cdot 16 - 5 \cdot 8 = 120.$$

Hence,

$$AD = \sqrt{120} = 2\sqrt{30}\,.$$

Solution 2:

By the angle bisector theorem, we get $BD = 5$. Let F be the foot of the altitude from D to AC.



By the angle bisector theorem, we get

$$BD = 5.$$

Let E be the foot of the perpendicular from A to BC. Let $ED = x$ and $AE = h$. Then

$$BE = 5 - x.$$

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Applying the Pythagorean Theorem to ΔABE and ΔACE, respectively, gives:

$$h^2 = 10^2 - (5-x)^2 = 16^2 - (8+x)^2.$$

Thus,

$$x = 4.5, \qquad h^2 = 10^2 - 0.5^2\,.$$

Now using the Pythagorean Theorem to ΔADE, we have:

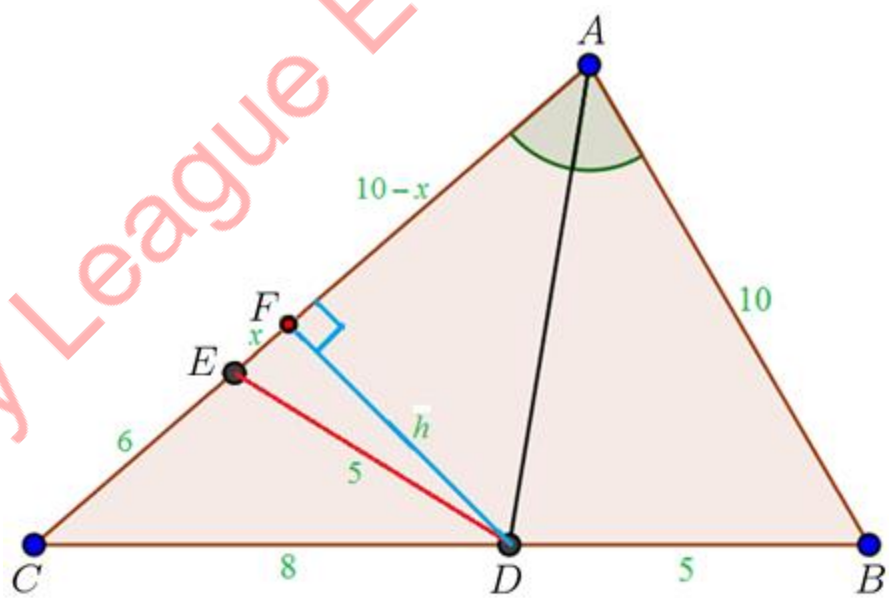
$$AD = \sqrt{x^2 + h^2} = \sqrt{4.5^2 + (10^2 - 0.5^2)} = \sqrt{120} = 2\sqrt{30}.$$

Solution 3:

By the angle bisector theorem, we get $DB = 5$. Take a point E on AC such that $AE = AB = 10$ and join DE. ΔADE is congruent to ΔADB by the SAS congruency criterion, and thus

$$DE = DB = 5.$$

Let F be the foot of the perpendicular from D to AC. Let $EF = x$ and $DF = h$.



Applying the Pythagorean Theorem to ΔDFE and ΔDFC, we get:

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$$h^2 = 5^2 - x^2 = 8^2 - (6 + x)^2.$$

Thus, $x = 0.25$. Using the Pythagorean Theorem to ΔADE and noting that $x^2 + h^2 = 25$, we have:

$$AD = \sqrt{(10 - x)^2 + h^2} = \sqrt{100 - 20x + (x^2 + h^2)}$$

$$= \sqrt{100 - 20 \times 0.25 + 25} = \sqrt{120} = 2\sqrt{30}.$$

Example 8: 2010 Princeton University Math Competition (PUMaC) Geometry A #3

Triangle ABC has $AB = 4, AC = 5,$ and $BC = 6$. An angle bisector is drawn from angle A, and meets BC at M. What is the nearest integer to $100\frac{AM}{CM}$?

Answer: 100

Solution:

By Angle-Bisector Theorem,

$$\frac{BM}{CM} = \frac{AB}{AC} = \frac{4}{5},$$

so

$$BM = \frac{8}{3}, \quad CM = \frac{10}{3}.$$

Using the formula for the length of an angle bisector, we have

$$AM = \sqrt{AB \cdot AC - BM \cdot CM} = \frac{10}{3}.$$

Hence,

$$\frac{AM}{CM} = 1.$$

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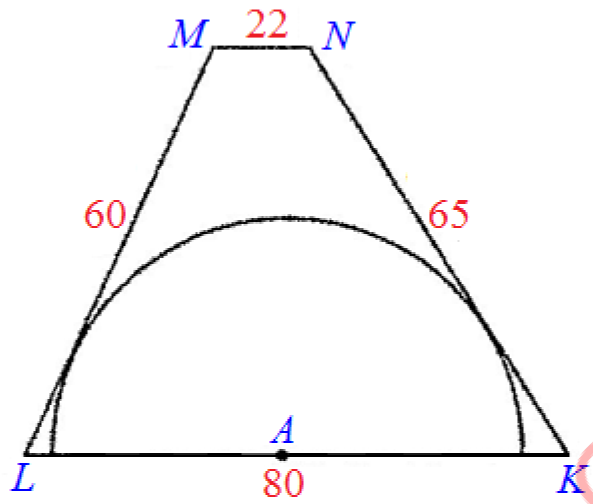
Example 9: 2013 MathCounts National Sprint #29

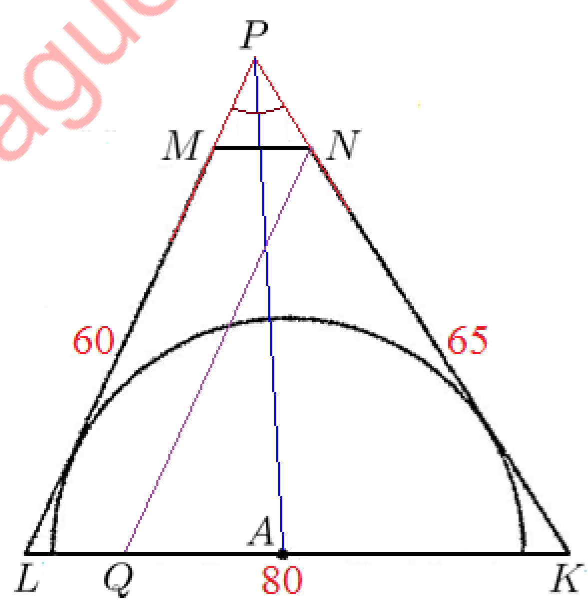


Trapezoid $KLMN$ has sides $KL = 80$ units, $LM = 60$ units, $MN = 22$ units, and $KN = 65$ units, with KL parallel to MN. A semicircle with center A on KL is drawn tangent to both sides KN and ML. What is the length of segment KA?

Express your answer as a mixed number.

Answer: $41\frac{3}{5}$

Solution:



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Extend LM and KN to intersect at P. From N draw a line parallel to LM to meet LK at Q. Then

$$QN = LM = 60.$$

By the AA similarity postulate,

$$\Delta LPK \sim \Delta QNK.$$

which implies that

$$\frac{KP}{LP} = \frac{KN}{QN} = \frac{65}{60} = \frac{13}{12}.$$

Recall that the line through an external point and the center of a circle bisects the angle formed by the two tangents from the external point. Thus, PA is the angle bisector of $\angle LPK$.

By the angle bisector theorem,

$$\frac{KA}{LA} = \frac{KP}{LP} = \frac{13}{12}.$$

Hence,

$$KA = \frac{13}{12+13} KL = \frac{13}{25} \cdot 80 = \frac{408}{5} = 41\frac{3}{5}.$$

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CLASSWORK

Exercise Set I

Exercise 1:

In ΔABC, $AB = 10$, and D is a point on side BC such that AD bisects $\angle BAC$. If $BD = 8$ and $CD = 4$, then what is the length of AC ?

(A) 4 (B) $\frac{9}{2}$ (C) 5 (D) $\frac{11}{2}$ (E) 6

Exercise 2:

In ΔABC, $\angle ABC = 30°$, D is the midpoint on AC such that BD is the bisector of $\angle ABC$. What is the degree measure of $\angle BAC$?

(A) 60° (B) 75° (C) 80° (D) 85° (E) 90°

Exercise 3:

In ΔABC, $AB = 10$, $BC = 8$, $AC = 12$. Let D be a point on side $\overline{AB}$ such that $\overline{CD}$ bisects $\angle C$. Then what is the length of $\overline{AD}$?

(A) $\frac{9}{2}$ (B) 5 (C) $\frac{11}{2}$ (D) 6 (E) 7

Exercise 4: 1989 ARML Team #4

In triangle ABC, angle bisectors AD and BE intersect at P. If $a = 3$, $b = 5$, $c = 7$, $BP = x$, and $PE = y$, compute the ratio $x : y$, where x and y are relatively prime integers.

Exercise 5: 1980 ARML Individual #1

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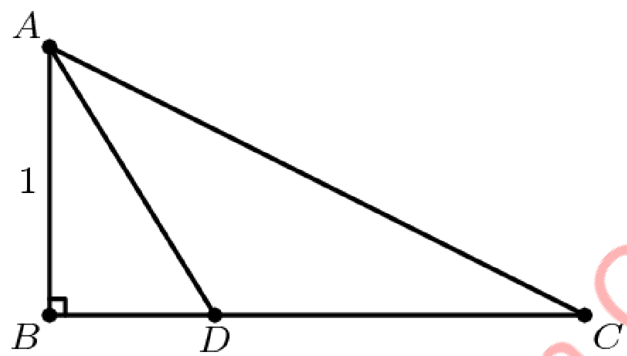
In ΔABC, the angle bisector AI divides the median BM into two segments of length 200 and 300, and AI divides BC into two segments of length 660 and x. Find the largest possible value of x.

Exercise 6: 2009 AMC 10B #20



Triangle ABC has a right angle at B, $AB = 1$, and $BC = 2$. The bisector of $\angle BAC$ meets $\overline{BC}$ at D. What is BD?

(A) $\frac{\sqrt{3}-1}{2}$ (B) $\frac{\sqrt{5}-1}{2}$ (C) $\frac{\sqrt{5}+1}{2}$ (D) $\frac{\sqrt{6}+\sqrt{2}}{2}$ (E) $2\sqrt{3}-1$

Exercise 7: 2010 AMC 10A #16/2010 AMC 12A #14

Nondegenerate ΔABC has integer side lengths, $\overline{BD}$ is an angle bisector, $AD = 3$, and $DC = 8$. What is the smallest possible value of the perimeter?

(A) 30 **(B)** 33 **(C)** 35 **(D)** 36 **(E)** 37

Exercise 8: 2014 AMC 10A #22

In rectangle $ABCD$, $\overline{AB} = 20$ and $\overline{BC} = 10$. Let E be a point on $\overline{CD}$ such that $\angle CBE = 15°$. What is $\overline{AE}$?

(A) $\frac{20\sqrt{3}}{3}$ **(B)** $10\sqrt{3}$ **(C)** 18 **(D)** $11\sqrt{3}$ **(E)** 20

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Exercise Set II

Exercise 1: 2004 AMC 10B #24

In triangle ABC we have $AB = 7$, $AC = 8$, and $BC = 9$. Point D is on the circumscribed circle of the triangle so that AD bisects angle BAC. What is the value of $\frac{AD}{CD}$?

(A) $\frac{9}{8}$ (B) $\frac{5}{3}$ (C) 2 (D) $\frac{17}{7}$ (E) $\frac{5}{2}$

Exercise 2: 1966 AHSME #11

The sides of triangle BAC are in the ratio $2:3:4$. BD is the angle-bisector drawn to the shortest side AC, dividing it into segments AD and CD. If the length of AC is 10, then the length of the longer segment of AC is:

(A) $3\frac{1}{2}$ (B) 5 (C) $5\frac{5}{7}$ (D) 6 (E) $7\frac{1}{2}$

Exercise 3: 1952 AHSME #19

Angle B of triangle ABC is trisected by BD and BE which meet AC at D and E, respectively. Then:

(A) $\frac{AD}{EC} = \frac{AE}{DC}$ **(B)** $\frac{AD}{EC} = \frac{AB}{BC}$ **(C)** $\frac{AD}{EC} = \frac{BD}{BE}$

(D) $\frac{AD}{EC} = \frac{(AB)(BD)}{(BE)(BC)}$ **(E)** $\frac{AD}{EC} = \frac{(AE)(BD)}{(DC)(BE)}$

Exercise 4: 2010 Stanford Math Tournament Geometry Test #4

Given triangle ABC. D lies on BC such that AD bisects $\angle BAC$. Given $AB = 3$, $AC = 9$, and $BC = 8$. Find AD.

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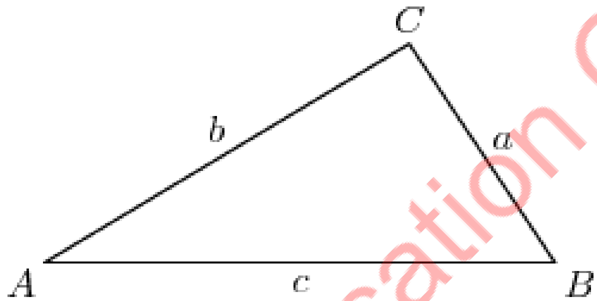
Exercise 5: 2009 AMC 12B #16

Trapezoid $ABCD$ has $AD||BC$, $BD = 1$, $\angle DBA = 23°$, and $\angle BDC = 46°$. The ratio $BC : AD$ is $9 : 5$. What is CD ?

(A) $\frac{7}{9}$ (B) $\frac{4}{5}$ (C) $\frac{13}{15}$ (D) $\frac{8}{9}$ (E) $\frac{14}{15}$

Exercise 6: 1985 AHSME #28

In ΔABC, we have $\angle C = 3\angle A$, $a = 27$, and $c = 48$. What is b?



(A) 33 (B) 35 (C) 37 (D) 39 (E) not uniquely determined

Exercise 7:

In ΔABC, $AB = 5$, $BC = 4$, $AC = 6$. Let D be a point on side $\overline{AB}$ such that $\overline{CD}$ bisects $\angle C$. Then what is the length of $\overline{CD}$?

(A) $\sqrt{5}$ (B) $2\sqrt{2}$ (C) $2\sqrt{3}$ (D) $3\sqrt{2}$ (E) $2\sqrt{5}$

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HOMEWORK

Problem Set I

Problem 1:

Let ABC be a triangle with angle bisector AD with D on BC. If $BD = 3, CD = 4$, and $AB + AC = 14$, what is $AC - AB$?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Problem 2:

In triangle $ABC, AB = 20$ and $AC = 10$. D is a point on BC such that $BD = \frac{20\sqrt{3}}{3}$ and $CD = \frac{10\sqrt{3}}{3}$. What is the measure of $\angle BAD - \angle CAD$ in degrees?

(A) 0 (B) 5 (C) 10 (D) 15 (E) 30

Problem 3:

In ΔABC, $AB = 7$, $AC = 14$ and D is a point on side BC such that AD bisects $\angle BAC$. If $BD = 5$, then what is the length of CD ?

(A) 8 (B) 9 (C) 10 (D) 12 (E) 14

Problem 4:

In ΔABC, AD is the bisector of $\angle A$ meeting side BC at D, if $AB = 10$, $AC = 14$, and $BC = 12$, then what is BD?

(A) 5 (B) $\frac{11}{2}$ (C) 6 (D) $\frac{2}{13}$ (E) 7

Problem 5:

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In triangle ABC, $AB = 13$, $BC = 14$, $AC = 15$. Let D denote the foot of the altitude from A to $\overline{BC}$ and let E denote the intersection of $\overline{BC}$ with the bisector of angle BAC. Which of the length of DE?

(A) 1 (B) $\frac{5}{4}$ (C) $\frac{3}{2}$ (D) 2 (E) $\frac{9}{4}$

Problem 6:

In ΔABC, $AC = BC$, and $\angle B = 72°$. Point D lies on BC such that AD bisects $\angle BAC$ and $CD = 1$. What is the length of BD?

(A) 1 (B) $\frac{1}{2}$ (C) $\frac{\sqrt{5}-1}{2}$ (D) $\frac{\sqrt{3}+1}{2}$ (E) $\frac{\sqrt{5}+1}{2}$

Problem 7:

In ΔABC, $AB = 4$, $BC = 6$, $AC = 5$. Let D be a point on side $\overline{AC}$ such that $\overline{BD}$ bisects $\angle B$. What is the length of $\overline{BD}$?

(A) $2\sqrt{2}$ (B) $2\sqrt{3}$ (C) $3\sqrt{2}$ (D) $2\sqrt{5}$ (E) $3\sqrt{3}$

Problem 8:

In ΔABC, $AB = 12$, $BC = 14$, $AC = 16$. The angle bisector of $\angle BAC$ intersects $\overline{BC}$ at D. What is the length of $\overline{AD}$?

(A) 12 (B) $\frac{25}{2}$ (C) $9\sqrt{2}$ (D) $8\sqrt{3}$ (E) 13

Problem 9:

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In ΔABC, $AB = 12$, $BC = 14$, $AC = 16$. The angle bisector of $\angle BAC$ intersects $\overline{BC}$ at D. The area of ΔABD can be written as $m\sqrt{n}$, where m and n are positive integers and m is not divisible by the square of any prime. What is $m + n$?

(A) 24 (B) 25 (C) 26 (D) 27 (E) 28

Problem 10: 1959 AHSME #28

In triangle ABC, AL bisects angle A and CM bisects angle C. Points L and M are on BC and AB, respectively. The sides of triangle ABC are $a, b,$ and c. Then

$$\frac{\overline{AM}}{\overline{MB}} = k\frac{\overline{CL}}{\overline{LB}}$$

where k is:

(A) 1 **(B)** $\frac{bc}{a^2}$ **(C)** $\frac{a^2}{bc}$ **(D)** $\frac{c}{b}$ **(E)** $\frac{c}{a}$

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Problem Set II

Problem 1: 1967 AHSME #21

In right triangle ABC the hypotenuse $\overline{AB}=5$ and leg $\overline{AC}=3$. The bisector of angle A meets the opposite side in A_1. A second right triangle PQR is then constructed with hypotenuse $\overline{PQ}=A_1B$ and leg $\overline{PR}=A_1C$. If the bisector of angle P meets the opposite side in P_1, the length of PP_1 is:

(A) $\frac{3\sqrt{6}}{4}$ **(B)** $\frac{3\sqrt{5}}{4}$ **(C)** $\frac{3\sqrt{3}}{4}$ **(D)** $\frac{3\sqrt{2}}{2}$ **(E)** $\frac{15\sqrt{2}}{16}$

Problem 2: 1975 AHSME #26

In acute $\triangle ABC$ the bisector of $\measuredangle A$ meets side BC at D. The circle with center B and radius BD intersects side AB at M; and the circle with center C and radius CD intersects side AC at N. Then it is always true that

(A) $\measuredangle CND+\measuredangle BMD-\measuredangle DAC=120^\circ$

(B) $AMDN$ is a trapezoid

(C) BC is parallel to MN

(D) $AM-AN=\frac{3(DB-DC)}{2}$

(E) $AB-AC=\frac{3(DB-DC)}{2}$

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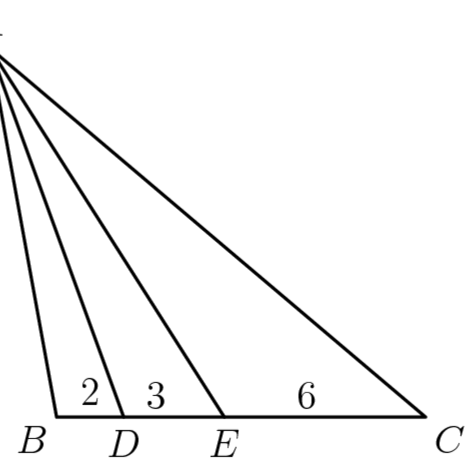
Problem 3: 1981 AHSME #25

In ΔABC in the adjoining figure, AD and AE trisect $\angle BAC$. The lengths of BD, DE, and EC are 2, 3, and 6, respectively. The length of the shortest side of ΔABC is



(A) $2\sqrt{10}$ **(B)** 11 **(C)** $6\sqrt{6}$ **(D)** 6 **(E)** not uniquely determined by the given information

Problem 4: 1983 AHSME #19

Point D is on side CB of triangle ABC. If $\angle CAD = \angle DAB = 60^\circ$, $AC = 3$ and $AB = 6$, then the length of AD is

(A) 2 (B) 2.5 (C) 3 (D) 3.5 (E) 4

Problem 5: 1998 AHSME #28

In triangle ABC, angle C is a right angle and $CB > CA$. Point D is located on $\overline{BC}$ so that angle CAD is twice angle DAB. If $AC/AD = 2/3$, then $CD/BD = m/n$, where m and n are relatively prime positive integers. Find $m + n$.

(A) 10 (B) 14 (C) 18 (D) 22 (E) 26

Problem 6: 2000 AMC 12 #17

A circle centered at O has radius 1 and contains the point A. The segment AB is tangent to the circle at A and $\angle AOB = \theta$. If point C lies on $\overline{OA}$ and $\overline{BC}$ bisects $\angle ABO$, then $OC =$

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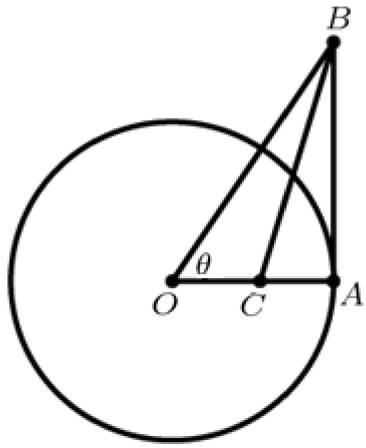

(A) $\sec^2\theta - \tan\theta$ (B) $\frac{1}{2}$ (C) $\frac{\cos^2\theta}{1+\sin\theta}$ (D) $\frac{1}{1+\sin\theta}$ (E) $\frac{\sin\theta}{\cos^2\theta}$

Problem 7: 2008 AMC 12A #20

Triangle ABC has $AC = 3$, $BC = 4$, and $AB = 5$. Point D is on $\overline{AB}$, and $\overline{CD}$ bisects the right angle. The inscribed circles of $\triangle ADC$ and $\triangle BCD$ have radii r_a and r_b, respectively. What is r_a/r_b?

(A) $\frac{1}{28}\left(10-\sqrt{2}\right)$ (B) $\frac{3}{56}\left(10-\sqrt{2}\right)$ (C) $\frac{1}{14}\left(10-\sqrt{2}\right)$ (D) $\frac{5}{56}\left(10-\sqrt{2}\right)$
(E) $\frac{3}{28}\left(10-\sqrt{2}\right)$

Problem 8: 2009 AMC 12B #16

Trapezoid $ABCD$ has $AD||BC$, $BD = 1$, $\angle DBA = 23^\circ$, and $\angle BDC = 46^\circ$. The ratio $BC : AD$ is $9 : 5$. What is CD?

(A) $\frac{7}{9}$ (B) $\frac{4}{5}$ (C) $\frac{13}{15}$ (D) $\frac{8}{9}$ (E) $\frac{14}{15}$

Problem 9: 2013 AMC 12B #24

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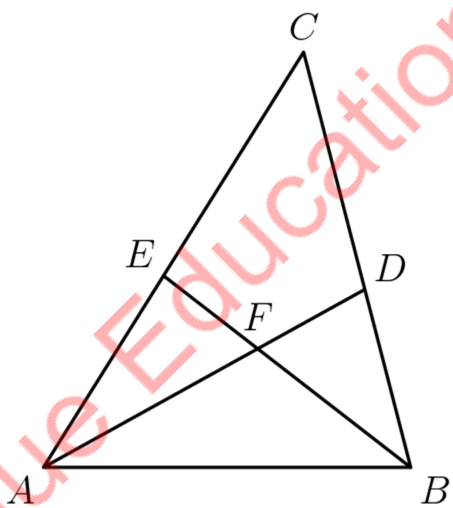
Let ABC be a triangle where M is the midpoint of $\overline{AC}$, and $\overline{CN}$ is the angle bisector of $\angle ACB$ with N on $\overline{AB}$. Let X be the intersection of the median $\overline{BM}$ and the bisector $\overline{CN}$. In addition $\triangle BXN$ is equilateral with $AC = 2$. What is BN^2?

$\textbf{(A)}\ \frac{10-6\sqrt{2}}{7} \qquad \textbf{(B)}\ \frac{2}{9} \qquad \textbf{(C)}\ \frac{5\sqrt{2}-3\sqrt{3}}{8} \qquad \textbf{(D)}\ \frac{\sqrt{2}}{6} \qquad \textbf{(E)}\ \frac{3\sqrt{3}-4}{5}$

Problem 10: 2016 AMC 12A #12

In $\triangle ABC$, $AB = 6$, $BC = 7$, and $CA = 8$. Point D lies on $\overline{BC}$, and $\overline{AD}$ bisects $\angle BAC$. Point E lies on $\overline{AC}$, and $\overline{BE}$ bisects $\angle ABC$. The bisectors intersect at F. What is the ratio $AF : FD$?



$\textbf{(A)}\ 3:2 \qquad \textbf{(B)}\ 5:3 \qquad \textbf{(C)}\ 2:1 \qquad \textbf{(D)}\ 7:3 \qquad \textbf{(E)}\ 5:2$

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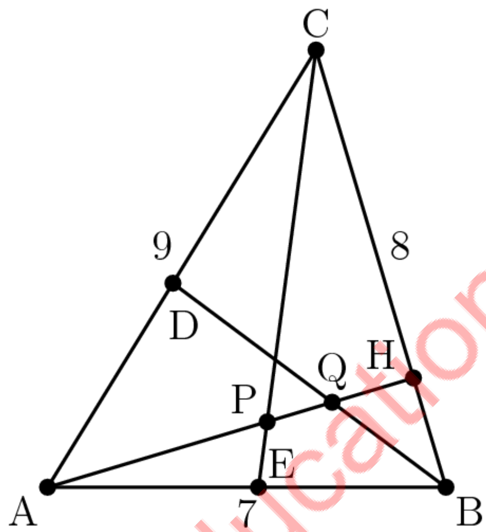
Problem Set III

Problem 1: 2016 AMC 12B #17

In $\triangle ABC$ shown in the figure, $AB = 7$, $BC = 8$, $CA = 9$, and $\overline{AH}$ is an altitude. Points D and E lie on sides $\overline{AC}$ and $\overline{AB}$, respectively, so that $\overline{BD}$ and $\overline{CE}$ are angle bisectors, intersecting $\overline{AH}$ at Q and P, respectively. What is PQ?



(A) 1 **(B)** $\frac{5}{8}\sqrt{3}$ **(C)** $\frac{4}{5}\sqrt{2}$ **(D)** $\frac{8}{15}\sqrt{5}$ **(E)** $\frac{6}{5}$

Problem 2: 2018 AMC 12B #12

Side $\overline{AB}$ of ΔABC has length 10. The bisector of angle A meets $\overline{BC}$ at D, and $CD = 3$. The set of all possible values of AC is an open interval (m, n). What is $m + n$?

(A) 16 **(B)** 17 **(C)** 18 **(D)** 19 **(E)** 20

Problem 3: 1980 AHSME #12

The equations of L_1 and L_2 are $y = mx$ and $y = nx$, respectively. Suppose L_1 makes twice as large of an angle with the horizontal (measured counterclockwise from the positive x-axis) as does L_2, and that L_1 has 4 times the slope of L_2. If L_1 is not horizontal, then mn is

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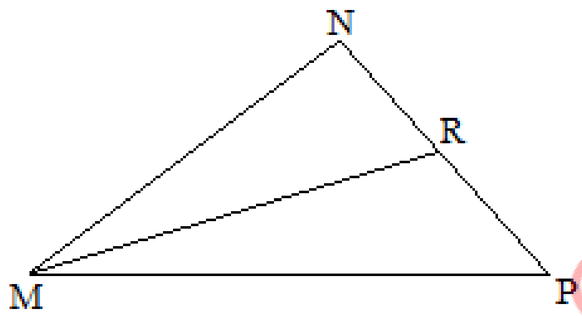
(A) $\frac{\sqrt{2}}{2}$ (B) $-\frac{\sqrt{2}}{2}$ (C) 2 (D) -2 (E) not uniquely determined

Problem 4: 2017 NC State Mathematics Finals: Level III #17



Consider the ΔMNP. $\overrightarrow{MR}$ bisects $\angle NMP$, $MN = 2y$, $NR = y$, $RP = y + 1$, and $MP = 3y - 1$. Find the value of y.

(A) 5 (B) 4 (C) 3 (D) 2 (E) 1

Problem 5: 2012 MathCounts National Team #10

What is the slope of the line that bisects the acute angle formed by $y = \frac{4}{3}x$ and $y = \frac{12}{5}x$? Express your answer as a common fraction.

Problem 6: 2010 MathCounts National Sprint #25

The sides of triangle CAB are in the ratio of $2 : 3 : 4$. Segment BD is the angle bisector drawn to the shortest side, dividing it into segments AD and DC. What is the length, in inches, of the longer subsegment of side AC if the length of side AC is 10 inches? Express your answer as a common fraction.

Problem 7: 2013 MathCounts State Target #8

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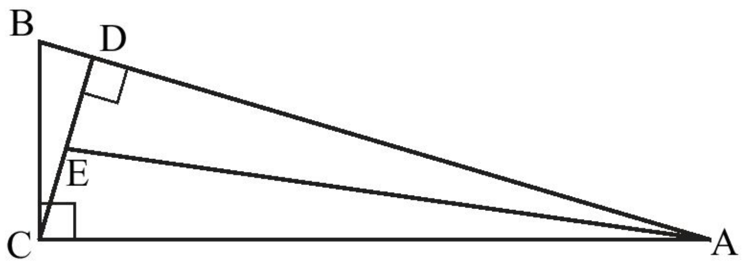
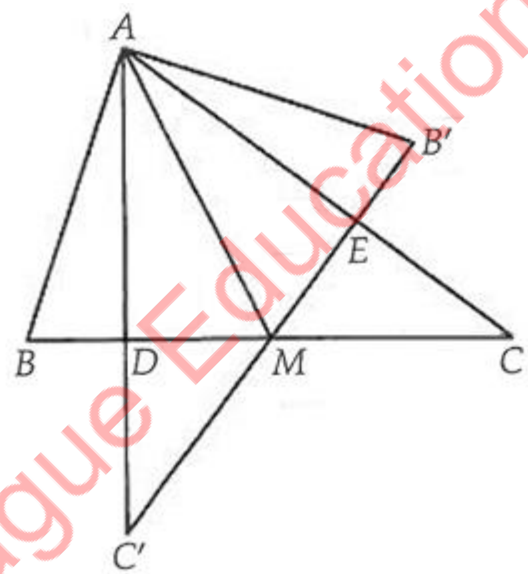


In right ΔABC, shown here, $AC = 24$ units and $BC = 7$ units. Point D lies on AB so that $CD \perp AB$. The bisector of the smallest angle of ΔABC intersects CD at point E. What is the length of ED? Express your answer as a common fraction.

Problem 8: 1987 ARML Individual Test #8



Triangle ABC is reflected in its median AM as shown. If $AE = 6,\ EC = 12,\ BD = 10$, and $AB = k\sqrt{3}$, compute k.

Problem 9: 2008 AMC 12A #20

Triangle ABC has $AC = 3$, $BC = 4$, and $AB = 5$. Point D is on $\overline{AB}$, and $\overline{CD}$ bisects the right angle. The inscribed circles of $\triangle ADC$ and $\triangle BCD$ have radii r_a and r_b, respectively. What is r_a/r_b?

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(A) $\frac{1}{28}\left(10-\sqrt{2}\right)$ (B) $\frac{3}{56}\left(10-\sqrt{2}\right)$ (C) $\frac{1}{14}\left(10-\sqrt{2}\right)$ (D) $\frac{5}{56}\left(10-\sqrt{2}\right)$
(E) $\frac{3}{28}\left(10-\sqrt{2}\right)$

Problem 10: 2017 CMIMC Geometry #3

In acute triangle ABC, points D and E are the feet of the angle bisector and altitude from A respectively. Suppose that

$$AC-AB=36 \text{ and } DC-DB=24.$$

Compute $EC-EB$.

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Problem Set IV (Bonus)

Problem 1: 2010 Princeton University Math Competition (PUMaC) Geometry A #3

Triangle ABC has $AB = 4, AC = 5$, and $BC = 6$. An angle bisector is drawn from angle A, and meets BC at M. What is the nearest integer to $100\frac{AM}{CM}$?

Problem 2: 2010 Stanford Math Tournament Geometry Test #4

Given triangle ABC. D lies on BC such that AD bisects $\angle BAC$. Given $AB = 3, AC = 9$, and $BC = 8$. Find AD.

Problem 3: 2018 CMIMC Geometry #2

Let $ABCD$ be a square of side length 1, and let P be a variable point on $\overline{CD}$. Denote by Q the intersection point of the angle bisector of $\angle APB$ with $\overline{AB}$. The set of possible locations for Q as P varies along $\overline{CD}$ is a line segment; what is the length of this segment?

Problem 4: 2016 CMIMC Geometry #3

Let ABC be a triangle. The angle bisector of $\angle B$ intersects AC at point P, while the angle bisector of $\angle C$ intersects AB at a point Q. Suppose the area of $\triangle ABP$ is 27, the area of $\triangle ACQ$ is 32, and the area of $\triangle ABC$ is 72. The length of $\overline{BC}$ can be written in the form $m\sqrt{n}$ where m and n are positive integers with n as small as possible. What is $m+n$?

Problem 5: 2014 Purple Comet math Meet (High School) #20

Triangle ABC has a right angle at C. Let D be the midpoint of side AC, and let E be the intersection of AC and the bisector of $\angle ABC$. The area of ΔABC is 144, and the area of ΔDBE is 8. Find AB^2.

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Problem 6: 2013 Harvard-MIT Math Tournament Guts Round #14

Consider triangle ABC with $\angle A = 2\angle B$. The angle bisectors from A and C intersect at D, and the angle bisector from C intersects AB at E. If

$$\frac{DE}{DC} = \frac{1}{3},$$

compute

$$\frac{AB}{AC}.$$

Problem 7: 2012 Harvard-MIT Math Tournament Team B #1

Triangle ABC has $AB = 5, BC = 3\sqrt{2}$, and $AC = 1$. If the altitude from B to AC and the angle bisector of angle A intersect at D, what is BD?

Problem 8: 2009 AIME I #5

Triangle ABC has $AC = 450$ and $BC = 300$. Points K and L are located on $\overline{AC}$ and $\overline{AB}$ respectively so that $AK = CK$, and $\overline{CL}$ is the angle bisector of angle C. Let P be the point of intersection of $\overline{BK}$ and $\overline{CL}$, and let M be the point on line BK for which K is the midpoint of $\overline{PM}$. If $AM = 180$, find LP.

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